THEORETICAL PHYSICS

General Invariance of Lagrangian Structures *)

by

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Summary. The lift of a local diffeomorphism of a manifold \( X \) to a prolongation \( Y \) of the bundle of frames of \( X \) is defined in terms of a functor between appropriate categories. A Lagrangian form defined on a bundle \( E \) associated to \( Y \) is said to be generally invariant if it is preserved by the lifts of all the local diffeomorphisms of the base space. It is shown that a generally invariant Lagrangian form is completely determined by a function on the typical fibre of \( E \).

1. Introduction. The notion of general invariance has been used and discussed since the advent of Einstein's relativistic theory of gravitation [1]. Hilbert analyzed the variational principles of classical physics and put forward the requirement of general invariance as a fundamental axiom [2]. The notion of invariance of a principle of least action may be conveniently defined when its Lagrangian is considered as a differential form on a fibre bundle [3—5]. In this paper, we develop the notions of differential geometry required to define precisely the concept of general invariance and we prove a theorem on the structure of generally invariant Lagrangians.

All the spaces and maps considered in this paper belong to the category of finite-dimensional, real differential manifolds of class \( C^\infty \). The subcategory of \( n \)-dimensional manifolds is denoted by \( D: f \in \text{Mor } D \) if and only if \( f \) is a diffeomorphism between \( n \)-dimensional manifolds. For any manifold \( X \) of dimension \( n \) there is the full subcategory \( D_X \) of \( D \) of all the local diffeomorphisms of \( X \) into itself. The category of principal bundles [6] over \( n \)-dimensional manifolds is denoted by \( PB \). A principal bundle is a triple \( (X, G, Y) \) of spaces, together with a pair \( (\pi, \delta) \) of maps, such that the Lie group \( G \) is the typical fibre of the bundle \( \pi: Y \to X, \delta: Y \times G \to Y \) defines a free action of \( G \) in \( Y \) on the right: \( \delta_a \circ \delta_b = \delta_{ab} \), where \( \delta_a \circ \delta_b = \delta \circ (y, a) \), \( y \in Y, a \in G \), and \( \pi \circ \delta = \pi \circ pr_1 \). A morphism of two principal bundles, \( (X_1, G_1, Y_1) \) and \( (X_2, G_2, Y_2) \), is a triple \( (f, g, h) \) of maps,

\[
f: X_1 \to X_2, \quad g: G_1 \to G_2, \quad h: Y_1 \to Y_2,
\]

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such that \( f \in \text{Mor } D \), \( g \) is a morphism of Lie groups, \( \delta_2 \circ (h \times g) = h \circ \delta_1 \), and \( \pi_2 \circ h = f \circ \pi_1 \).

The **frame functor** \( F: D \rightarrow PB \) associates to a manifold \( X \) the principal bundle \((X, \text{GL}(n, R), F_0X)\) of frames, \( EF = (f, \text{id}_{\text{GL}(n, R)}, F_0 f) \), where \( F_0 f \) is the map of frames induced by the diffeomorphism \( f \). The tangent functor is denoted by \( T \). The **Cartan (contravariant) functor** which associates to \( X \) the exterior algebra of fields of differential forms over \( X \) is denoted by a star. Thus

\[
X^* = \bigoplus_{p=0}^{n} X_p^*,
\]

where \( X_p^* \) is the module of \( p \)-forms (\( X_0^* \) is simply the algebra of differentiable functions on \( X \)). If \( f: X \rightarrow Y \) and \( a \in Y^* \), then \( f^* a \in X^* \) is the pull-back of \( a \) by \( f \) (if \( a \in Y_0^* \), then \( f^* a = a \circ f \)).

If \( \pi_E: E \rightarrow X \) is a bundle, then a \( p \)-form \( a \) on \( E \) is said to be **horizontal** relative to \( \pi_E \) if

\[
u \perp a = 0 \quad \text{for any} \quad u \in TE \quad \text{such that} \quad T\pi_E(u) = 0.
\]

Let \( \theta_x = (\theta^i_x) \), \( i = 1, \ldots, n \), be the canonical, \( R^n \)-valued 1-form on \( F_0X \) [7]. The \( n \)-form

\[
\mu_x = \theta^1_x \wedge \theta^2_x \wedge \cdots \wedge \theta^n_x
\]

is horizontal relative to the natural projection of \( F_0X \) on \( X \). Since the base manifold is usually fixed, it is convenient to write \( \mu \) instead of \( \mu_x \), and this will be done. If \( \delta \) defines the action of \( \text{GL}(n, R) \) in \( F_0X \), then, for any \( a \in \text{GL}(n, R) \),

\[
(1) \quad \delta_a^* \mu = (\det a)^{-1} \mu.
\]

Moreover,

\[
(2) \quad (F_0f)^* \mu = \mu
\]

for any \( f \in \text{Mor } D_X \).

2. Lifting.

**Definition.** A covariant functor \( \tau: D \rightarrow PB \) is said to define a **lifting** to the Lie group \( G \) if

\[
\tau f = (f, \text{id}_G, \tau_0 f), \quad \text{for any } f \in \text{Mor } D,
\]

and there exists a natural transformation \( N \) from \( \tau \) to \( F \) such that

\[
N(X) = (\text{id}_X, g_X, i_X) \quad \text{for any } X \in \text{Ob } D.
\]

The isomorphism of bundles \( \tau_0 f: \tau_0 X_1 \rightarrow \tau_0 X_2 \) is called the **lift** of \( f: X_1 \rightarrow X_2 \). A lifting is said to be **transitive** if the lifts act transitively on \( Y = \tau_0 X \), i.e., if for any \( y_1, y_2 \in Y \) there exists \( f \in \text{Mor } D_X \) such that \( (\tau_0 f)(y_1) = y_2 \). For example, the bundle of holonomic frames of order \( q \) is obtained by a transitive lifting to the group \( G^q(n) \) [8]. The bundle of affine frames [7] of a manifold is obtained by a non-transitive lifting to the affine group.
3. Lagrangians and invariance. Let \((X, G, Y)\) be a principal bundle and let 
\(\sigma: G \times Z \to Z\) be a map defining the action of \(G\) in \(Z\) on the left, \(\sigma_a \circ \sigma_b = \sigma_{ab}\), where 
\(\sigma_a(z) = \sigma(a, z)\), \(a \in G\), \(z \in Z\). The action of \(G\) may be extended to \(Y \times Z\) by putting 
\(\psi_a(y, z) = (\delta_a(y), \sigma_{a^{-1}}(z))\). The quotient space 
\[ E = (Y \times Z)/G \]
can be made into a bundle over \(X\), \(\pi_E: E \to X\), with \(\pi_E \circ k = \pi \circ \text{pr}_1\), where \(k: Y \times Z \to E\) is the canonical map, 
\[ k(y, z) = k(y', z') \quad \text{iff} \quad \text{there exists} \quad a \in G \quad \text{such that} \quad (y', z') = \psi_a(y, z). \]

The notion of a Lagrangian structure on \(\pi_E\) over an \(n\)-dimensional base may be defined in two equivalent ways:

I. By giving an \(n\)-form *) \(\lambda\) on \(E\), horizontal relative to \(\pi_E\).

II. By giving an \(n\)-form \(\overline{\lambda}\) on \(Y \times Z\), horizontal relative to \(\pi \circ \text{pr}_1\) and invariant under the action of \(G\),

\[ \psi_a^* \overline{\lambda} = \overline{\lambda} \quad \text{for any} \quad a \in G. \]

If either one of these two forms is given, the other may be obtained from the formula:

\[ k^* \lambda = \overline{\lambda}. \]

If \(U \subset X\) is a relatively compact open set and \(s: U \to E\) is a local section of \(\pi_E\), then the number

\[ \Lambda(s) = \int_U s^* \lambda \]
is the action of \(s\) corresponding to \(\lambda\). Let \((f, \text{id}_G, h)\) be an automorphism of \((X, G, Y)\) and let \((f, h_E)\) be the corresponding automorphism of the associated bundle \(\pi_E\):

\[ h_E \circ k = k \circ (h \times \text{id}), \quad \pi_E \circ h_E = f \circ \pi_E. \]

We say that \(h_E\) is induced from \(h\) by \(\sigma\). The Lagrangian structure on \(\pi_E\) is said to be invariant with respect to \(h\) if

\[ h_E^* \lambda = \lambda \]
or, equivalently, if

\[ (h \times \text{id})^* \overline{\lambda} = \overline{\lambda}. \]

The automorphism \(h\) defines a permutation \(H\) of the set of local sections of \(\pi_E\),

\[ H(s) = h_E \circ s \circ f^{-1}, \]

and the invariance of the Lagrangian is equivalent to that of the action,

\[ \Lambda \circ H = \Lambda. \]

*) To be precise, we should have assumed that \(\lambda\) is an odd form or that \(X\) is endowed with a preferred orientation. This would have resulted in inessential changes in the paper.
4. General invariance. Consider a Lagrangian structure defined by a horizontal n-form $\lambda$ on the bundle $\pi_E: E \to X$ associated to a principal bundle $(X, G, Y)$ obtained by a lifting $\tau$ of $X$ to $G$, $Y = \tau_0 X$. If $f \in \text{Mor} \, D_X$, we write $f_E$ to denote the local diffeomorphism of $E$ induced from $\tau_0 f$ by $\sigma$. The base space being now fixed, the morphism of principal bundles defined by the natural transformation $N: \tau \to F$ may be written as $N(X) = (\text{id}, g, j)$, where $g: G \to \text{GL}(n, R)$ is a morphism of Lie groups and $j: Y \to F_0 X$ is such that:

\begin{align*}
(4) & \quad j \circ \delta_a = \delta_{a(a)} \circ j \quad \text{for any } a \in G, \\
(5) & \quad j \circ \tau_0 f = (F_0 f) \circ j \quad \text{for any } f \in \text{Mor} \, D_X.
\end{align*}

The $n$-form $j^* \mu$ on $Y$ is horizontal relative to $\pi: Y \to X$ and we can write

$$\bar{\lambda}(y, z) = L(y, z) \cdot (i^* \mu)(y),$$

or

$$\bar{\lambda} = L \cdot (j \circ \text{pr}_1)^* \mu,$$

where, as before, $\text{pr}_1: Y \times Z \to Y$ is the first projection. This defines the Lagrange function $L: Y \times Z \to R$, satisfying

$$L \circ \psi_a = \det g(a) \cdot L, \quad \text{for any } a \in G,$$

by virtue of (1), (3) and (4). Clearly, a Lagrangian structure on a bundle associated to a bundle obtained by lifting may be defined also in terms of such a Lagrange function $L$.

**Definition I.** The Lagrangian structure defined by $\bar{\lambda}$ on $\pi_E: E \to X$ is **generally invariant** if

$$f_E^* \bar{\lambda} = \bar{\lambda} \quad \text{for any } f \in \text{Mor} \, D_X.$$

Clearly, there is an equivalent

**Definition II.** The Lagrangian structure is generally invariant if

\begin{equation}
(6) \quad (\tau_0 f \times \text{id})^* \bar{\lambda} = \bar{\lambda} \quad \text{for any } f \in \text{Mor} \, D_X.
\end{equation}

The main result of this paper is contained in the

**Theorem.** A Lagrangian structure on a bundle associated to a principal bundle obtained by a transitive lifting is generally invariant if and only if the corresponding Lagrange function $L$ does not depend on the first argument. Any function $\mathcal{L}: Z \to R$ such that

\begin{equation}
(7) \quad \mathcal{L} \circ \sigma_{a^{-1}} = \det g(a) \cdot \mathcal{L}
\end{equation}

defines a generally invariant Lagrangian structure on the bundle associated by $\sigma$ to the principal bundle which results by lifting.

**Proof.** Consider the Lagrangian structure defined by $L$. The requirement of invariance (6) leads to

\begin{equation}
(8) \quad L \cdot (j \circ \text{pr}_1)^* \mu = L \circ (\tau_0 f \times \text{id}) \cdot (j \circ (\tau_0 f) \circ \text{pr}_1)^* \mu.
\end{equation}
From (2) and (5) we obtain

\[(j \circ \tau_0 f)^* \mu = j^* \mu.\]

Eq. (8) reduces to \(L(y, z) = L((\tau_0 f)(y), z)\) and implies

\[L(y_1, z) = L(y_2, z), \quad \text{for any } y_1, y_2 \in Y,\]

if the lifting is transitive. Conversely, if \(\mathcal{L}: Z \to R\) is \(g\)-equivariant [9], i.e., if it satisfies (7), then \(L = \mathcal{L} \circ \text{pr}_2: Y \times Z \to R\) defines a generally invariant Lagrangian structure irrespectively of whether the lifting is transitive or not.

In this manner, the question of general invariance of a variational principle is reduced to the problem of equivariance of the corresponding Lagrange function with respect to a (finitesimal) Lie group \(G\). This result should be compared and contrasted with the classical approach to generally invariant variational problems [10].

REFERENCES