BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences math., astr.
et phys. — Vol. XXII, No. 2, 1974

THEORETICAL PHYSICS

General Invariance of Lagrangian Structures *)

by

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Presented by A. TRAUTMAN on November 12, 1973

Summary. The lift of a local diffeomorphism of a manifold X to a prolongation Y of the bundle of frames of X is defined in terms of a functor between appropriate categories. A Lagrangian form defined on a bundle E associated to Y is said to be generally invariant if it is preserved by the lifts of all the local diffeomorphisms of the base space. It is shown that a generally invariant Lagrangian form is completely determined by a function on the typical fibre of E.

1. Introduction. The notion of general invariance has been used and discussed since the advent of Einstein's relativistic theory of gravitation [1]. Hilbert analyzed the variational principles of classical physics and put forward the requirement of general invariance as a fundamental axiom [2]. The notion of invariance of a principle of least action may be conveniently defined when its Lagrangian is considered as a differential form on a fibre bundle [3—5]. In this paper, we develop the notions of differential geometry required to define precisely the concept of general invariance and we prove a theorem on the structure of generally invariant Lagrangians.

All the spaces and maps considered in this paper belong to the category of finite-dimensional, real *differential manifolds* of class C^∞. The subcategory of n-dimensional manifolds is denoted by $\boldsymbol{D}$: $f \in \operatorname{Mor} \boldsymbol{D}$ if and only if f is a diffeomorphism between n-dimensional manifolds. For any manifold X of dimension n there is the full subcategory $\boldsymbol{D}_X$ of $\boldsymbol{D}$ of all the local diffeomorphisms of X into itself. The category of *principal bundles* [6] over n-dimensional manifolds is denoted by $\boldsymbol{PB}$. A principal bundle is a triple (X, G, Y) of spaces, together with a pair (π, δ) of maps, such that the Lie group G is the typical fibre of the bundle $\pi: Y \to X$, $\delta: Y \times G \to Y$ defines a free action of G in Y on the right: $\delta_a \circ \delta_b = \delta_{ba}$, where $\delta_a(y) = \delta(y, a)$, $y \in Y$, $a \in G$, and $\pi \circ \delta = \pi \circ pr_1$. A morphism of two principal bundles, (X_1, G_1, Y_1) and (X_2, G_2, Y_2), is a triple (f, g, h) of maps,

$$f: X_1 \to X_2, \quad g: G_1 \to G_2, \quad h: Y_1 \to Y_2,$$

*) This paper was written during the first author's stay at the Institute of Theoretical Physics, Warsaw University. The research of the second author was supported in part by the National Science Foundation.

such that $f \in \operatorname{Mor} \boldsymbol{D}$, g is a morphism of Lie groups, $\delta_2 \circ (h \times g) = h \circ \delta_1$, and $\pi_2 \circ h = f \circ \pi_1$.

The *frame functor* $F: \boldsymbol{D} \to \boldsymbol{PB}$ associates to a manifold X the principal bundle $(X, \mathrm{GL}(n, \boldsymbol{R}), F_0 X)$ of frames, $Ff = (f, \mathrm{id}_{\mathrm{GL}(n, \boldsymbol{R})}, F_0 f)$, where $F_0 f$ is the map of frames induced by the diffeomorphism f. The tangent functor is denoted by T. The *Cartan* (contravariant) *functor* which associates to X the exterior algebra of fields of differential forms over X is denoted by a star. Thus

$$X^* = \bigoplus_{p=0}^{n} X_p^*,$$

where X_p^* is the module of p-forms (X_0^* is simply the algebra of differentiable functions on X). If $f: X \to Y$ and $a \in Y^*$, then $f^* a \in X^*$ is the pull-back of a by f (if $a \in Y_0^*$, then $f^* a = a \circ f$).

If $\pi_E: E \to X$ is a bundle, then a p-form a on E is said to be *horizontal* relative to π_E if

$$u \lrcorner a = 0 \quad \text{for any} \quad u \in TE \quad \text{such that} \quad T\pi_E(u) = 0.$$

Let $\theta_X = (\theta_X^i)$, $i = 1, \ldots, n$, be the canonical, $\boldsymbol{R}^n$-valued 1-form on $F_0 X$ [7]. The n-form

$$\mu_X = \theta_X^1 \wedge \theta_X^2 \wedge \ldots \wedge \theta_X^n$$

is horizontal relative to the natural projection of $F_0 X$ on X. Since the base manifold is usually fixed, it is convenient to write μ instead of μ_X, and this will be done. If δ defines the action of $\mathrm{GL}(n, \boldsymbol{R})$ in $\mathrm{F}_0 X$, then, for any $a \in \mathrm{GL}(n, \boldsymbol{R})$,

$$\delta_a^* \mu = (\det a)^{-1} \mu . \tag{1}$$

Moreover,

$$(F_0 f)^* \mu = \mu \tag{2}$$

for any $f \in \operatorname{Mor} \boldsymbol{D}_X$.

2. Lifting.

Definition. A covariant functor $\tau: \boldsymbol{D} \to \boldsymbol{PB}$ is said to define a *lifting* to the Lie group G if

$$\tau f = (f, \mathrm{id}_G, \tau_0 f), \text{ for any } f \in \operatorname{Mor} \boldsymbol{D},$$

and there exists a natural transformation N from τ to F such that

$$N(X) = (\mathrm{id}_X, g_X, j_X) \quad \text{for any } X \in \operatorname{Ob} \boldsymbol{D}.$$

The isomorphism of bundles $\tau_0 f: \tau_0 X_1 \to \tau_0 X_2$ is called the *lift* of $f: X_1 \to X_2$. A lifting is said to be *transitive* if the lifts act transitively on $Y = \tau_0 X$, i.e., if for any $y_1, y_2 \in Y$ there exists $f \in \operatorname{Mor} \boldsymbol{D}_X$ such that $(\tau_0 f)(y_1) = y_2$. For example, the bundle of holonomic frames of order q is obtained by a transitive lifting to the group $G^q(n)$ [8]. The bundle of affine frames [7] of a manifold is obtained by a non-transitive lifting to the affine group.

3. Lagrangians and invariance. Let (X, G, Y) be a principal bundle and let $\sigma\colon G\times Z\to Z$ be a map defining the action of G in Z on the left, $\sigma_a\circ\sigma_b=\sigma_{ab}$, where $\sigma_a(z)=\sigma(a, z)$, $a\in G$, $z\in Z$. The action of G may be extended to $Y\times Z$ by putting $\psi_a(y, z)=\big(\delta_a(y), \sigma_{a^{-1}}(z)\big)$. The quotient space

$$E=(Y\times Z)/G$$

can be made into a bundle over X, $\pi_E\colon E\to X$, with $\pi_E\circ k=\pi\circ\mathrm{pr}_1$, where $k\colon Y\times Z\to E$ is the canonical map,

$$k(y, z)=k(y', z') \quad \text{iff there exists} \quad a\in G \quad \text{such that} \quad (y', z')=\psi_a(y, z).$$

The notion of a *Lagrangian structure* on π_E over an n-dimensional base may be defined in two equivalent ways:

I. By giving an n-form*) λ on E, horizontal relative to π_E.

II. By giving an n-form $\bar\lambda$ on $Y\times Z$, horizontal relative to $\pi\circ\mathrm{pr}_1$ and invariant under the action of G,

$$\psi_a^*\bar\lambda=\bar\lambda \quad \text{for any } a\in G. \tag{3}$$

If either one of these two forms is given, the other may be obtained from the formula:

$$k^*\lambda=\bar\lambda.$$

If $U\subset X$ is a relatively compact open set and $s\colon U\to E$ is a local section of π_E, then the number

$$A(s)=\int_U s^*\lambda$$

is the *action* of s corresponding to λ. Let (f, id_G, h) be an automorphism of (X, G, Y) and let (f, h_E) be the corresponding automorphism of the associated bundle π_E:

$$h_E\circ k=k\circ(h\times\mathrm{id}), \qquad \pi_E\circ h_E=f\circ\pi_E.$$

We say that h_E is *induced* from h by σ. The Lagrangian structure on π_E is said to be *invariant* with respect to h if

$$h_E^*\lambda=\lambda$$

or, equivalently, if

$$(h\times\mathrm{id})^*\bar\lambda=\bar\lambda.$$

The automorphism h defines a permutation H of the set of local sections of π_E,

$$H(s)=h_E\circ s\circ f^{-1},$$

and the invariance of the Lagrangian is equivalent to that of the action,

$$A\circ H=A.$$

*) To be precise, we should have assumed that λ is an odd form or that X is endowed with a preferred orientation. This would have resulted in inessential changes in the paper.

4. General invariance. Consider a Lagrangian structure defined by a horizontal n-form λ on the bundle $\pi_E\colon E\to X$ associated to a principal bundle (X, G, Y) obtained by a lifting τ of X to G, $Y=\tau_0 X$. If $f\in \operatorname{Mor} \boldsymbol{D}_X$, we write f_E to denote the local diffeomorphism of E induced from $\tau_0 f$ by σ. The base space being now fixed, the morphism of principal bundles defined by the natural transformation $N\colon \tau\to F$ may be written as $N(X)=(\mathrm{id}, g, j)$, where $g\colon G\to \mathrm{GL}(n, \boldsymbol{R})$ is a morphism of Lie groups and $j\colon Y\to F_0 X$ is such that:

$$j\circ\delta_a=\delta_{g(a)}\circ j \quad \text{for any } a\in G, \tag{4}$$

$$j\circ\tau_0 f=(F_0 f)\circ j \quad \text{for any } f\in \operatorname{Mor} \boldsymbol{D}_X. \tag{5}$$

The n-form $j^*\mu$ on Y is horizontal relative to $\pi\colon Y\to X$ and we can write

$$\bar{\lambda}(y, z)=L(y, z)\cdot(i^*\mu)(y),$$

or

$$\bar{\lambda}=L\cdot(j\circ \mathrm{pr}_1)^*\mu,$$

where, as before, $\mathrm{pr}_1\colon Y\times Z\to Y$ is the first projection. This defines the *Lagrange function* $L\colon Y\times Z\to \boldsymbol{R}$, satisfying

$$L\circ\psi_a=\det g(a)\cdot L, \quad \text{for any } a\in G,$$

by virtue of (1), (3) and (4). Clearly, a Lagrangian structure on a bundle associated to a bundle obtained by lifting may be defined also in terms of such a Lagrange function L.

DEFINITION I. The Lagrangian structure defined by λ on $\pi_E\colon E\to X$ is *generally invariant* if

$$f_E^*\lambda=\lambda \quad \text{for any } f\in \operatorname{Mor} \boldsymbol{D}_X.$$

Clearly, there is an equivalent

DEFINITION II. The Lagrangian structure is generally invariant if

$$(\tau_0 f\times \mathrm{id})^*\bar{\lambda}=\bar{\lambda} \quad \text{for any } f\in \operatorname{Mor} \boldsymbol{D}_X. \tag{6}$$

The main result of this paper is contained in the

THEOREM. *A Lagrangian structure on a bundle associated to a principal bundle obtained by a transitive lifting is generally invariant if and only if the corresponding Lagrange function L does not depend on the first argument. Any function* $\mathcal{L}\colon Z\to \boldsymbol{R}$ *such that*

$$\mathcal{L}\circ\sigma_{a^{-1}}=\det g(a)\cdot\mathcal{L} \tag{7}$$

defines a generally invariant Lagrangian structure on the bundle associated by σ *to the principal bundle which results by lifting.*

Proof. Consider the Lagrangian structure defined by L. The requirement of invariance (6) leads to

$$L\cdot(j\circ \mathrm{pr}_1)^*\mu=L\circ(\tau_0 f\times \mathrm{id})\cdot(j\circ(\tau_0 f)\circ \mathrm{pr}_1)^*\mu. \tag{8}$$

From (2) and (5) we obtain

$$(j \circ \tau_0 f)^* \mu = j^* \mu .$$

Eq. (8) reduces to $L(y, z) = L\big((\tau_0 f)(y), z\big)$ and implies

$$L(y_1, z) = L(y_2, z), \quad \text{for any } y_1, y_2 \in Y,$$

if the lifting is transitive. Conversely, if $\mathcal{L}: Z \to \boldsymbol{R}$ is g-equivariant [9], i.e., if it satisfies (7), then $L = \mathcal{L} \circ \mathrm{pr}_2 : Y \times Z \to \boldsymbol{R}$ defines a generally invariant Lagrangian structure irrespectively of whether the lifting is transitive or not.

In this manner, the question of general invariance of a variational principle is reduced to the problem of equivariance of the corresponding Lagrange function with respect to a (finitedimensional) Lie group G. This result should be compared and contrasted with the classical approach to generally invariant variational problems [10].

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Д. Крупка, А. Траутман, **Общая инвариантность структур Лагранжа**

Содержание. Поднятие локального диффеоморфизма многообразия X до продолжения Y пучка реперов этого многообразия, было сформулировано при помощи соответствующего функтора. Форма Лагранжа, определенная на пучке E, связанном с Y является обще-инвариантной, если она сохранена через поднятия всех локальных диффеоморфизмов базисного пространства. В работе показано, что обще-инвариантная форма Лагранжа определяется полностью некоторой функцией на типовом волокне пучка E.