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THE EINSTEIN-CARTAN THEORY OF GRAVITATION

by Andrzej TRAUTMAN
Instytut Fizyki Teoretycznej Uniwersytetu Warszawskiego
00-681 Warszawa, ul. Hoza 69

RESUME

En 1923, E. Cartan a proposé une modification de la théorie de la relativité générale d'Einstein. Dans la théorie modifiée, on admet comme modèle d'Univers une variété différentiable munie d'un tenseur métrique et d'une connexion linéaire euclidienne asymétrique. La torsion de cette connexion est directement liée à la densité du spin. L'article contient quelques résultats récents obtenus par l'auteur dans le cadre de la théorie d'Einstein et de Cartan.

INTRODUCTION

The properties of gravitational waves propagating in empty space are fairly well understood. Their classical behaviour is adequately described by the field equations of Einstein's theory of general relativity. Much less is known about the interaction of gravitational radiation with matter, and very little about the quantum effects of gravity. According to J. Weber, there is more gravitational radiation of cosmic origin than expected on the ground of computations based on Einstein's equations and our present knowledge of the distribution of matter in the Galaxy. The rate of emission of gravitational waves depends on the precise form of the field equations and on the way in which matter acts as the source of the field. There is an interesting modification of Einstein's theory which affects the form of the equations only inside matter. The modification, due to Elie Cartan, gives rise to a new, relativistic theory which we propose to call the Einstein-Cartan theory of gravitation. The purpose of the report is to present a brief account of the theory and its application to cosmology.

A heuristic motivation for considering the modified theory may be found elsewhere [1]-[4] and a good presentation of its history and earlier results is given in a series of papers by Hehl [5], [6], [21], which also contain a comprehensive bibliography. This report is restricted to statements of definitions and results without proofs.

DEFINITIONS AND NOTATION

The **Einstein-Cartan theory** is a classical relativistic theory of space, time, and gravitation. As a model of space-time it assumes a four-dimensional differential manifold X with a metric tensor g and a linear connection ω. Following Lichnerowicz, the manifold, the metric, and the linear connection are assumed to be of class (C_2, C_4 piecewise), (C_1, C_3 piecewise), and (C_0, C_2 piecewise), respectively. The metric has a normal hyperbolic signature and is compatible with the linear connection (in other words, the connection is "metric" or "Euclidean").

Locally, the manifold always admits a field of co-frames (θ^i), $i = 1, 2, 3, 4$, and $g = g_{ij}\theta^i \otimes \theta^j$. Relative to (θ^i), the linear connection is represented by the 1-forms ω^i_j which may be used to compute the 2-forms of curvature :

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j = \frac{1}{2}\theta^k \wedge R^i_{jk} = \frac{1}{2} R^i_{jkl}\, \theta^k \wedge \theta^l,$$

and torsion

$$\Theta^i = d\theta^i + \omega^i_j \wedge \theta^j = \frac{1}{2}\theta^j \wedge Q^i_j = \frac{1}{2} Q^i_{jk}\, \theta^j \wedge \theta^k.$$

The linear connection defines on X a covariant exterior derivative D which generalizes both the covariant derivative ∇ and the exterior differential d ; e.g., $\Theta^i = D\theta^i$ and $Dg_{ij} = dg_{ij} - \omega_{ij} - \omega_{ji}$, where $\omega_{ij} = g_{ik}\omega^k_j$. The curvature and torsion forms satisfy the Bianchi identities

$$D\,\Omega^i_j = 0 \qquad \text{and} \qquad D\,\Theta^i = \Omega^i_j \wedge \theta^j.$$

The Levi-Civita pseudotensor $\eta_{ijkl} = \eta_{[ijkl]}$, $\eta_{1234} = |\det g|^{1/2}$, gives rise to a collection of forms,

$$\eta_{ijk} = \theta^l\, \eta_{ijkl} \quad , \quad \eta_{ij} = \frac{1}{2}\theta^k \wedge \eta_{ijk}$$

$$\eta_i = \frac{1}{3}\theta^j \wedge \eta_{ij}, \quad \eta = \frac{1}{4}\theta^i \wedge \eta_i ,$$

the last of which is the volume element of X. The metric condition $Dg_{ij} = 0$ implies

$$D\eta_{ijkl} = 0 \quad , \quad D\eta_{ijk} = \Theta^l\, \eta_{ijkl}$$

$$D\eta_{ij} = \Theta^k \wedge \eta_{ijk} = (Q^k_{ij} - \delta^k_i Q^l_{lj} - \delta^k_j Q^l_{il})\, \eta_k$$

$$D\eta_i = \Theta^j \wedge \eta_{ij} = Q^j_{ij}\, \eta.$$

UNIQUENESS OF THE GRAVITATIONAL LAGRANGIAN

Similarly as in Einstein's theory, the field equations may be derived from a principle of least action consisting of a gravitational and a matter part. The gravitational part in essentially unique because of (*)

Theorem 1. Any pseudo 4-form on X which is intrinsic and linear homogeneous in (Ω, Θ) is proportional to

$$8\pi K = \frac{1}{2} \eta_{ij} \wedge \Omega^{ij}.$$

By varying the metric, the frames, and the linear connection independently of one another, one obtains

$$8\pi\delta K = \frac{1}{2} \eta E^{ij} \delta g_{ij} + \delta\theta^i \wedge e_i - \frac{1}{2} \delta\omega^i_j \wedge c^j_i + \text{an exact form},$$

where E^{ij} is the (generalized) Einstein tensor,

$$e_i = \frac{1}{2} \eta_{ijk} \wedge \Omega^{jk} \qquad \text{and} \qquad c_{ij} = - D\eta_{ij}.$$

Theorem 2. The following identity holds [7]

$$\eta E^{ij} = \theta^j \wedge e^i - \frac{1}{2} D c^{ij}.$$

Theorem 3. For any metric tensor g, the equation $c_{ij} = 0$ is equivalent to

$$\omega^i_j = \gamma^i_j + \delta^i_j \lambda$$

where γ is the Riemannian connection associated to g and λ is a 1-form on X.

THE FIELD EQUATIONS

The metric and the linear connection are determined in the Einstein-Cartan theory by the system of equations

$$e_i = -\frac{8\pi G}{c^4} t_i \tag{1}$$

(*) I owe this theorem to a remark of J. Ehlers.

$$c_{ij} = -\frac{8\pi G}{c^4} s_{ij} \tag{2}$$

$$Dg_{ij} = 0 \tag{3}$$

where t_i and $s_{ij} = -s_{ji}$ are 3-forms describing the sources of the field. If one defines a symmetric tensor T^{ij} by

$$\eta T^{ij} = \theta^j \wedge t^i - \frac{1}{2} Ds^{ij}$$

then the system (1), (2), (3) may be replaced by an equivalent system of equations, containing

$$E^{ij} = -\frac{8\pi G}{c^4} T^{ij}$$

instead of eq. (1). The physical interpretation of the sources may be inferred from the covariant conservation laws which follow from the Bianchi identities :

Theorem 4. The field equations (1)-(3) imply [8]

$$Dt_i = Q_i^j \wedge t_j - \frac{1}{2} R_i^{kl} \wedge s_{kl} \tag{4}$$

$$Ds_{ij} = \theta_j \wedge t_i - \theta_i \wedge t_j \tag{5}$$

The equations of Einstein's theory are obtained from (1)-(3) by putting $s_{ij} = 0$. In this case, torsion vanishes, $\eta T^{ij} = \theta^j \wedge t^i$, and (4) reduces to the covariant conservation law of energy-momentum, $Dt_i = 0$.

In the limit of special relativity, $G \to 0$, the connection is integrable and has no torsion. The Cartesian coordinates satisfy $Dx^i = \theta^i$, energy-momentum is conserved,

$$Dt_i = 0$$

and eq. (5) may be written as

$$D(x^i t^j - x^j t^i + s^{ij}) = 0.$$

Therefore, the 3-form (s^{ij}) may be interpreted as the density of **spin,** whereas (t^i) is the density of **energy and momentum.**

SYMMETRIES AND CONSERVATION LAWS

The transposed connection $\widetilde{\omega}$ is defined by

$$\widetilde{\omega}^i_j = \omega^i_j + Q^i_j$$

and may be used to compute the covariant exterior derivative $\widetilde{D}$ relative to $\widetilde{\omega}$. A **symmetry** (automorphism) of space-time in the Einstein-Cartan theory is a diffeomorphism of X which preserves both g and ω.

Consider a one-parameter group of transformations of X generated by the vector field v. A necessary and sufficient condition for the transformations to be symmetries is that the Lie derivatives of g and ω with respect to v vanish [11] :

$$\widetilde{\nabla}^i v^j + \widetilde{\nabla}^j v^i = 0 \tag{6}$$

$$D\widetilde{\nabla}_j v^i + v \lrcorner \Omega^i_j = 0, \tag{7}$$

In a Riemannian space, the connections ω and $\widetilde{\omega}$ coincide and (7) is a consequence of the Killing (6).

Similarly as in Einstein's theory of general relativity, symmetries of space-time give rise to conservation laws in the form of an "ordinary divergence", $dj = 0$. The conservation theorem may be obtained from the covariant conservation laws given by (4) and (5). A simple computation leads to

Theorem 5. If v generates a one-parameter group of symmetries of X, then there holds the conservation law [9]

$$dj = 0$$

for the current j defined by

$$j = v^i t_i + \frac{1}{2}\, \widetilde{\nabla}^k v^l s_{kl} \tag{9}$$

EQUATIONS OF MOTION

Let (u^i) be a velocity field, i.e., a smooth vector field on X, normalized by $g_{ij}u^i u^j = 1$. Consider the 3-form $u = u^i \eta_i$ and define, for any tensor field (φ_A) on X, its **particle derivative** $(\dot{\varphi}_A)$ relative to u [12] :

$$\dot{\varphi}_A\, \eta = D(\varphi_A\, u).$$

Following Weyssenhoff and Raabe [13], a **spinning dust** may be defined as a continuous medium characterized by its velocity (u^i), the density of energy and momentum (P_i), and the density of spin (S_{ij}). The 3-forms of energy-momentum and of spin are

$$t_i = \mathrm{P}_i u \qquad \text{and} \qquad s_{ij} = \mathrm{S}_{ij} u, \tag{10}$$

respectively. From eq. (5) there follows

$$\mathrm{P}^i = \rho u^i - u_k \dot{\mathrm{S}}^{ki} \tag{11}$$

where

$$\rho = g_{ij} \mathrm{P}^i u^j.$$

Eq. (5) is equivalent to the system consisting of eq. (11) and the equation of motion of spin

$$\dot{\mathrm{S}}^{ij} = u^i u_k \dot{\mathrm{S}}^{kj} - u^j u_k \dot{\mathrm{S}}^{ki} \tag{12}$$

Eq. (4) gives rise to the equation of translatory motion [10]

$$\dot{\mathrm{P}}_i = \left(\mathrm{Q}^k_{ij} \mathrm{P}_k - \frac{1}{2} \mathrm{R}^{kl}_{ij} \mathrm{S}_{kl} \right) u^j, \tag{13}$$

which is a generalization, to the Einstein-Cartan theory, of an equation derived by Mathisson [14] and Papapetrou [15] for point particles with an intrinsic angular momentum. It is easy to prove

Theorem 6. A spinless test particle moves along a geodesic of the Riemannian connection associated with g (even if X torsion).

A SIMPLE, NON-SINGULAR COSMOLOGICAL MODEL WITH SPIN

Some time ago, I conjectured that the singularities of gravitational collapse and cosmology may be averted by the direct influence of spin on geometry, as taken into account in the Einstein-Cartan theory [16].

Recently, Kopczyński [17] constructed a class of non-singular cosmological models based on the Einstein-Cartan theory of gravitation. The models provide a lower bound for the minimum radius of the world, corresponding to the hypotetical bounce, presumably occurring during the hot stage of the development of the Universe. One of the simplest models is for a Universe filled with a spinning dust characterized by its four-dimensional velocity (u^i), density of mass ρ, and density of spin $s_{ij} = \mathrm{S}_{ij} u$, $u^j \mathrm{S}_{ij} = 0$. These assumptions are compatible with a Robertson-Walker line-element ds,

$$ds^2 = c^2\, dt^2 - \mathrm{R}(t)^2\, (dx^2 + dy^2 + dz^2)$$

and $u^i = \delta^i_4$, with $x^1 = x$, $x^2 = y$, $x^3 = z$, and $x^4 = \mathrm{ct}$. Assuming that the spins are aligned along the x-axis, $\mathrm{S}_{23} = \sigma$ and $\mathrm{S}_{jk} = 0$ for $j\,.\,k \neq 6$,

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DISCUSSION

TRAUTMAN/DESER

There is no conflict between the derivation here presented and the ECT, because its net effect is a direct (contact) spin-spin coupling, which, like all other material interactions is of course permitted. The ECT does not imply the existence of a "second gravitation", which would indeed be a discrepancy. I also empharize that the classical GR breaks down necessarily at particle compton radii ($\sim 10^{-13}$ cm), long before the Planck length (10^{-33} cm).

Finally it is to be recalled that in the derivation leading to GR, one consequence is that the orbital and spin parts of the $T_{\mu\nu}$ matrix elements are necessarily coupled with the same strength, so that there is also a nice place for spin also in normal Einstein theory.

TRAUTMAN/STEWART

1) Peter Hajicek and I have made a calculation of the amount of anisotropy permitted in the Universe which would allow a singularity to be averted. We

find that a singularity could only be avoided, if the fractional anisotropy in the Hubble constant is :

$$\frac{\Delta H}{H} \lesssim 10^{-81}$$

The best observational evidence is :

$$\frac{\Delta H}{H} \lesssim 10^{-6}$$

2) In order to formulate an experiment which would compare the GR and Einstein-Cartan theories one probably needs to consider a possibly rotating fluid. The total angular velocity vector is then the sum of the vorticity and the spin, and so one wants an experiment which differentiates between angular velocity and vorticity.

TRAUTMAN/Z. PERJES

To what extent will this theory modify the validity of various unicity theorems and conjectures of black hole physics ? Does it seem possible to verify experimentally the predictions of the theory under appropriate circumstances ? The ring and internal satellites of Saturn, for instance, feel a considerably high relative angular momentum.

Is there any reason to believe that spins were correlated during the hottest stage of the evolution of the Universe, say for $|t| \lesssim \tau$? A mechanism is provided by a cosmic magnetic field H, since, if it exists at all, it may successfully compete against increasing temperature T, provided that the flux is conserved, $HR^2 = \text{const.}$ Since $TR = \text{const.}$, the ratio $\mu H/kT$ behaves like $1/R$ and may have been so large in the past as to have allowed the magnetic field to align all spins. More precisely, if μ is the nuclear magneton and H_0 is the (unknown) present value of the intergalactic magnetic field, then $\mu H(t)/kT(t) \approx 10^{20}\ H_0/R(t)$ where H_0 is measured in gauss and $R(t)$ in centimeters.

The model described here suffers from at least two defects : It neglects both the pressure and the magnetic field energy in the description of the hot stage of the development of the Universe. It would be interesting to know whether a closed cosmological model would also bounce due to torsion. An indication that it would indeed is provided by a solution of the Einstein-Cartan equations with an inhomogeneous, spherically, symmetric distribution of spins where the Friedmann singularity is again averted [20], [21] although the curvature is not regular. It may also be expected that torsion due to the increasing alignment of spins in a collapsing magnetic star will prevent the occurrence of singularities, even after the star has crossed its event horizon.

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and making use of equation (2) to determine the torsion, one can reduce (1) to a modified Friedmann equation

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} + \frac{3G^2S^2}{2c^4R^4} = 0 \qquad (14)$$

supplemented by the conservation laws of mass and spin,

$$M = \frac{4}{3}\pi\rho R^3 = \text{const.} \quad \text{and} \quad S = \frac{4}{3}\pi\sigma R^3 = \text{const.}$$

The last term on the left side of equation (14) plays the role of a "repulsive potential" which is effective at small values of R and prevents the solution from ever approaching zero. In fact, the equation may be solved exactly,

$$R(t) = R_{min}(1 + t^2/\tau^2)^{1/3} \quad \text{where} \quad \tau = S/\sqrt{3}\,Mc^2$$

and

$$R(0) = R_{min} = (3GS^2/2Mc^4)^{1/3}.$$

At $t = 0$, the radius of a sphere containing N particles of mass $m = M/N$ and spin $1/2\,\hbar = S/N$ is

$$R_{min} = (3NG\hbar^2/8mc^4)^{1/3}.$$

If m is the mass of a neutron, then

$$R_{min} \approx 3.10^{-27}\,N^{1/3}\ \text{cm}.$$

For $N \approx 10^{80}$, a figure which is often quoted as representing the total number of baryons in the part of the Universe accessible to observation, R_{min} is of the order of one centimeter. This may appear to be a rather small size for the Universe, but it is very large when compared to the Planck length $(G\hbar/c^3)^{1/2} \approx 1{,}6.10^{-33}$ cm which has been considered as providing the only natural limitation on the validity of classical Einsteinian cosmology [18], [19]. In our model, the density of matter at $t = 0$ is of the order of $m^2c^4/G\hbar^2 \approx 10^{55}$ gcm^{-3}. It is much smaller than the density $c^5/G^2\hbar \approx 10^{94}$ gcm^{-3} at which the quantum effects of the gravitational field are presumed to play a dominant role. At $t = 0$, the torsion is of the same order of magnitude as the inverse Compton wavelength of the particles. For $t \gg \tau = \hbar/mc^2 \approx 10^{-23}$ sec the model coincides with the corresponding solution of Einstein's equations. Therefore any considerations concerning the hadronic era of the Universe, and its subsequent development, will be little affected by the introduction of torsion in cosmology.