30.

8

Theory of Gravitation

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1. INTRODUCTION

The work on, and the understanding of, gravitation greatly influenced not only the physicist's conception of nature but also the development of all exact sciences. Newton invented the method of fluxions, and thereby laid down the foundations of calculus, in connection with his research on the motion of bodies and on the law of universal attraction.[1] The calculus of variations, the theory of differential equations and the perturbation methods of solving them arose directly from the needs of mechanics and astronomy. Through the work of Poincaré[2], the consideration of global and stable properties of motions stimulated the birth of topology. The relativistic theory of gravitation of Einstein[3], and his search for a unified theory[4], enhanced the development of differential geometry. The notion of a superspace introduced recently by J. A. Wheeler[5] provides us with a concrete example of an infinite-dimensional manifold and leads to a number of difficult problems in the theory of Banach manifolds.

The theory of gravitation has had successes in all the fields where gravitational interactions are expected to play a dominant role. The laws of gravitation, very accurately checked within the solar system, seem to be applicable also on a much larger scale. It is amazing – and encouraging – that a simple theory of gravitation provides us with models of the entire Universe, some of which are at least in a qualitative agreement with the observations.

The achievements of the Newtonian theory of gravitation were later overshadowed by those of Maxwell's electromagnetic theory, by the discovery of the atomic nature of matter and by the development of quantum mechanics and relativity. The theory of general relativity although initially poor in experimentally verifiable predictions, greatly influenced our picture of the Universe and the understanding of space and time. It also gave rise to a hope – which is now believed to be false – of constructing a unified, geometric theory of electromagnetism and gravitation. In spite of its profound implications, for a long time Einstein's theory was being developed with little contact with the natural sciences. The situation has changed during the last years, thanks to the startling discoveries in astronomy, the progress in radio and radar measurements and the patient efforts to detect gravitational waves.[6] The theorists have followed suit and done relevant work on the process of collapse and formation of black holes, on new general relativistic effects, on the mechanisms of emission and absorption of

J. Mehra (ed.), The Physicist's Conception of Nature, 179-198.

gravitational radiation, and on the stability of relativistic, gravitating systems. The significance of the new discoveries and observations, as well as the role of general relativity in astrophysics, astronomy and cosmology, have been admirably presented in the lectures of S. Chandrasekhar and D. W. Sciama which are printed in the same volume.

Much excitement and justified interest surrounds the experiments performed with the purpose of measuring the flux of *gravitational waves* falling on the surface of the Earth. According to J. Weber, who initiated this field of research over a decade ago, there are sporadic pulses of radiation which seem to come from the centre of the Galaxy.[7] This result would constitute a beautiful confirmation of Einstein's predictions if it were not for the fact that its interpretation in terms of gravitational waves requires the existence of extremely powerful, hard-to-find sources of radiation. Although the issue is important and interesting, it is more relevant for astrophysics and cosmology than for the theory of gravitation as such. If Weber is right, then we are faced with the challenge to find the sources of the powerful radiation; if he is not, then this only confirms the earlier, conservative estimates of the amount of gravitational radiation in the Universe. In the latter case, more refined techniques than those available now will be needed to detect gravitational waves of cosmic origin. In either case, there does not seem to be any need for a change in the fundamental assumption of the general theory of relativity. Moreover, the recent accurate measurements of the time delay of radar signals passing near the surface of the Sun, and also those of the deflection of radio waves, seem to confirm the theory fairly well and favour the conventional theory rather than its modifications, such as those requiring an additional flat metric or a scalar field.[8]

An outstanding problem of theoretical physics is to build a *quantum theory of space, time, and gravitation.* For brevity, the problem is often formulated by stating that the gravitational field should be quantized. Such a description is not entirely adequate because it presupposes a quantum theory of gravity along the lines of quantum electrodynamics. Gravitation is so closely related to the structure of space-time that it is hard to conceive a profound modification of the description of the former without introducing drastic changes in the nature of the latter.

A pioneering work on the quantum theory of gravity was done in 1930 by L. Rosenfeld and a satisfactory Hamiltonian form of Einstein's equations was given by P. A. M. Dirac. Extensive research on various methods of quantizing general relativity and on possible quantum effects of gravitation has been carried out since 1950.[9,10] Assuming that it is correct to describe gravitational interactions in terms of their quanta, the main quantitative result is that gravitons may produce observable effects only at extremely high energies, corresponding to the *Planck length*

$$(\hbar G/c^3)^{1/2} \approx 10^{-33}\,\text{cm}. \tag{1}$$

For example, according to L. Parker[11], R. U. Sexl and H. K. Urbantke[12], Ya. B. Zeldovich and his co-workers[13], one can expect creation of pairs of particles by very strong gravitational fields, with curvatures of the order of $10^{33}\,\text{cm}^{-1}$. It is presumed that such curvatures may occur during gravitational collapse, be it cosmological or local. However, as S. Hawking and R. Penrose[14] point out, and A. Salam predicts on the basis of his theory, extraordinary local effects should take place already at curvatures of the order of 10^{13} cm^{-1}. It is difficult to imagine how a particle such as an electron, whose radius is of the order of 10^{-13} cm, can survive in a space with a (local) radius of curvature 10^{20} times smaller. A different point of view, advanced by J. A. Wheeler, is to consider elementary particles as having a foam-like structure, the foam consisting of highly curved, quantum-fluctuating space-time with a characteristic length given by Equation (1).

The theoretical studies indicate the importance of the Planck length but so far there is no experimental evidence that this quantity is physically relevant in a similar sense as the fine structure constant, the classical radius of the electron, the Chandrasekhar mass or the gravitational radius of the Sun are known to be. In other words, can we be confident that nothing drastic happens when we consider the range of distances from 10^{-13} cm down to 10^{-33} cm (or rather the corresponding range of energies)? In this unexplored region there may occur completely new phenomena which will eventually mask over the quantum gravitational effects, as calculated from the present theory.

A short lecture on a broad subject cannot be comprehensive. It would not be appropriate to review here the fundamentals of the theory of gravitation. The article by J. Mehra, appearing in this volume, contains a lucid account of how the modern theory of gravitation was created, with an emphasis on the role played by Hilbert. In this lecture, I shall restrict myself to a few basic problems connected with the development of general relativity theory and to the Einstein–Cartan theory of gravitation.

2. THE PRINCIPLES

The principles which are associated with the theory of relativity and gravitation were the subject of many controversies and misunderstandings. One of the best known among them has been the discussion on the significance of the 'principle of general covariance', a polemic which started around 1917[15] and has been revived during the recent years by V. A. Fock.[16]

In part, the difficulties are due to a lack of clarity as to what is a principle in theoretical physics. We accept the following definition: a *principle* is a statement about physical theories, formulated on the basis of experiments or by extrapolation from known theories. If the principle is a true statement for any particular theory, then the theory is said to satisfy the principle. As a rule, a principle is meaningful (true or false) for a class of physical theories and not for all theories. In other words, a principle selects a set of theories, namely those for which it is true. These remarks may sound trivial but, if accepted, they show that such familiar arguments as 'it follows from the principle of equivalence alone that light propagating in a gravitational field changes its frequency' cannot stand good. It requires a definite physical theory to investigate the propagation of light. The principle of equivalence is satisfied by several theories, including the Newtonian theory of gravitation[17], which does not allow any reasonable description of electromagnetism.

To illustrate the general definition by a well-understood example, let us consider the *principle of (special) relativity*. It refers to the class of theories which assume an affine (flat) space as a model of space-time. Each theory is characterized by some additional non-dynamical structure (integrable linear connection, metric tensor, absolute time, ether, etc.). Moreover, free motions of point particles are described by a family of straight lines in the affine space, and this family is an open and non-void subset of the space of all straight lines. With every theory there is associated a group of automorphisms: it is the group of all these affine transformations which preserve the additional structure of space-time. The principle of relativity says that the group of automorphisms acts transitively in the family of free motions. Clearly, the principle of relativity, as defined here, is satisfied in Einstein's special theory and in Galilean physics, fails in pre-relativistic electrodynamics, and is meaningless in theories based on a curved space-time.

The *principle of equivalence* refers to classical theories of gravitation which assume an affinely connected space-time (i.e., a differentiable manifold with a linear connection). The principle says that, in the vacuum, the geometry of space-time defines locally only one linear connection. It implies that there is really no such thing as a gravitational force. Indeed, a force is described by a vector field which can be used to build a new linear connection from any given one, in contradiction with the principle of equivalence. Therefore, the equality of inertial and gravitational masses is a consequence of the principle.

I am tempted at this point to make the following remark which goes a little beyond the subject of my talk. As soon as one realizes that the gravitational force is not a correct concept, it becomes clear that any classical force, everything that can be legitimately put on the right-hand side of Newton's law of motion, is of electromagnetic origin. On the other hand, most of theoretical physics, except for general relativity but including quantum theory, is based on concepts such as the energy, a Hamiltonian or a Lagrangian, which all can be traced back to the notion of force. We are not so naive as to try to reduce all phenomena to electromagnetism, as were the nineteenth-century physicists with respect to 'mechanical forces' but we attempt to model all theories after electrodynamics, classical or quantum. It may be that this is one of the reasons of the slow progress in our understanding of the fundamental processes.

In order to formulate the *principle of general invariance* (or the principle of general relativity as it is sometimes called), it is desirable to distinguish between the dynamical and the absolute (non-dynamical) elements of a theory.[18,19] The dynamical elements characterize the history of a physical system described by the theory and are subject to equations of motion. In any given theory, the absolute elements are the same for all histories. For example, the metric tensor is an absolute element in the theory of special relativity and acquires a dynamical character in Einstein's theory of gravitation. The automorphisms or symmetries of a theory are the transformations which preserve the absolute. According to the principle of general invariance, the automorphism group of a relativistic theory of gravitation consists of all diffeomorphisms of space-time. This is a highly non-trivial and strong statement; it has nothing to do with the possibility of going over to curvilinear coordinates.

The following is a list of the absolute elements in the known classical theories, supplemented by a conjecture about the future theory of space, time, and gravitation:

Theory of	Absolute elements		
	Time	Metric and flat linear connection	Topology and differential structure
Galilean mechanics	yes	yes	yes
special relativity	no	yes	yes
general relativity	no	no	yes
the future	no	no	no

The topological and differential structures of space-time do not seem to possess a well-defined operational meaning. Therefore, it is likely that they will have to be abandoned, or rather replaced by another structure which will be more closely related to, and influenced by, physical phenomena than the absolute, locally Euclidean manifold structure of space-time assumed in all current theories. In my opinion, a satisfactory quantum theory of space, time, and gravitation will have to do away with the notion of a differentiable manifold as a model of space-time.

The *principle of locality* in classical physics can also be precisely stated in the language of differential geometry. Roughly, it says that all fundamental laws of physics can be reduced to equations involving only local differential operators of finite order. The *principle of Mach* is a negation of the principle of locality. Of course, there is nothing unique or final about the formulation of any of the principles. The definitions given here should be considered as tentative examples of how the subject can be approached. In particular, Mach's principle ought to be sharpened to become significant.

3. CATEGORIES, FIBRE BUNDLES AND GAUGE INVARIANCE

The purpose of a physical theory is to construct mathematical models of nature, models that can be used to explain and predict physical phenomena and events. Any particular theory, perhaps with the exception of cosmology, provides us with many models, each of them adapted to a specific situation and giving a good description of events within a bounded region of space and time and with an accuracy characteristic of the theory. The details change from one model to another but all models of a theory have certain common features, determined by the basic assumptions of the theory. This remark leads at once to the idea that it should be possible to organize the mathematical models used in a physical theory into what is nowadays called a category.[20]

To establish the terminology and notation, let me recall that a *category* $\mathscr{A}$ consists of a class of *objects* $A, B, C, \ldots$, and a class of sets $\mathrm{Mor}(A, B)$, $\mathrm{Mor}(B, C), \ldots$ of elements called *morphisms* of $\mathscr{A}$. If $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then there exists the composite morphism $g \circ f \in \mathrm{Mor}(A, C)$ and the composition of morphisms is associative. For any object B there is a morphism $1_B \in \mathrm{Mor}\ (B, B)$ such that

$$\text{if} \quad f \in \mathrm{Mor}(A, B) \quad \text{and} \quad g \in \mathrm{Mor}(B, C), \quad \text{then} \quad 1_B \circ f = f \quad \text{and} \quad g \circ 1_B = g.$$

In most cases, morphisms are certain mappings and one writes

$$f: A \to B \quad \text{instead of} \quad f \in \mathrm{Mor}(A, B).$$

A morphism $f: A \to B$ is called an *isomorphism* if there exists a morphism $f^{-1}: B \to A$, called its inverse, such that

$$f \circ f^{-1} = 1_B \quad \text{and} \quad f^{-1} \circ f = 1_A.$$

From any category $\mathscr{A}$ one can form the category $\mathscr{I}\mathscr{A}$ whose objects coincide with those of $\mathscr{A}$ and whose morphisms are isomorphisms of $\mathscr{A}$.

For example, the category of sets, $\mathscr{E}ns$, has sets as objects and mappings as morphisms; bijections are isomorphisms. The category of (real) vector spaces $\mathscr{V}ect$ has linear mappings as morphisms. In physics, of importance is the category $\mathscr{D}iff$ of finite-dimensional differential manifolds, with differentiable mappings as morphisms, and the category $\mathscr{A}ff$ of affine spaces, which may be considered as a subcategory of $\mathscr{D}iff$. Let $A_1 = (E_1, V_1, +)$ and $A_2 = (E_2, V_2, +)$ be two affine spaces, where E_1 and E_2 are the underlying sets, V_1 and V_2 are the associated vector spaces and $+$ in both cases denotes the transitive and free action of the additive groups V_1 and V_2 in E_1 and E_2, respectively. By definition, a morphism in $\mathscr{A}ff$ is a map $f: E_1 \to E_2$ such that there exists a linear map $\tau f: V_1 \to V_2$ and

$$f(v + p) = \tau f(v) + f(p)$$

for any $p \in E_1$ and $v \in V_1$. If $g: E_2 \to E_3$ is another affine morphism, then

$$\tau(g \circ f) = \tau g \circ \tau f.$$

We have here an example of a correspondence between categories, referred to as a *functor*.

More generally, if $\mathscr{A}$ and $\mathscr{B}$ are categories, a law τ associating to objects A, B and morphisms f, g of $\mathscr{A}$ certain objects and morphisms of $\mathscr{B}$, and such that

$$\begin{gathered} \text{if} \quad f: A \to B, \quad \text{then} \quad \tau(f): \tau(A) \to \tau(B), \\ \tau(1_A) = 1_{\tau(A)}, \qquad \tau(g \circ f) = \tau(g) \circ \tau(f) \end{gathered}$$

is called a covariant functor. (Here 1_A denotes the identity morphism of A.) A contravariant functor $\tau:\mathscr{A}\to\mathscr{B}$ is characterized by

$$\text{if} \quad f:A\to B, \quad \text{then} \quad \tau(f):\tau(B)\to\tau(A),$$
$$\tau(1_A)=1_{\tau(A)} \quad \text{and} \quad \tau(g\circ f)=\tau(f)\circ\tau(g).$$

Clearly, $\tau:\mathscr{A}\mathit{ff}\to\mathscr{V}\mathit{ect}$ defined, in the notation of the previous paragraph, by

$$\tau(A)=E, \qquad \tau(f)=\tau f$$

is a covariant functor.

For any category $\mathscr{A}$ and any object C in $\mathscr{A}$ one defines the contravariant functor

$$\tau^C:\mathscr{A}\to\mathscr{E}\mathit{ns}$$

by

$$\tau^C(A)=\mathrm{Mor}(A,C)$$

and

$$(\tau^C f)(g)=g\circ f$$

for any $f:A\to B$ and $g:B\to C$. For example, if $\mathscr{A}$ is the category of vector spaces and $C=R$ then τ^C associates to any vector space V its dual V^*; this defines a contravariant functor $*:\mathscr{V}\mathit{ect}\to\mathscr{V}\mathit{ect}$.

One of the important applications of categories and functors is to define the concept of naturality. Given two categories, $\mathscr{A}$ and $\mathscr{B}$, and two functors of the same variance $\tau_1,\tau_2:\mathscr{A}\to\mathscr{B}$, a natural transformation N associates to each object A of $\mathscr{A}$ a morphism in $\mathscr{B}$, $N(A):\tau_1(A)\to\tau_2(A)$ such that the diagram

$$\begin{tikzcd}
\tau_1(A) \arrow[r, "\tau_1(f)"] \arrow[d, "N(A)"'] & \tau_1(B) \arrow[d, "N(B)"] \\
\tau_2(A) \arrow[r, "\tau_2(f)"'] & \tau_2(B)
\end{tikzcd}$$

commutes for any $f:A\to B$. If $N(A)$ is an isomorphism for any A, one says that N establishes a *natural equivalence* of the functors τ_1 and τ_2. For example, let $\mathscr{A}=\mathscr{B}=\mathscr{V}\mathit{ect}$ and $\tau_1=id$, $\tau_2=**$ (double dual). The mapping $N(V):V\to V^{**}$, defined for any vector space V by

$$\langle v^*, N(V)v\rangle=\langle v, v^*\rangle, \qquad v\in V, \quad v^*\in V^*,$$

is a natural transformation. It becomes a natural equivalence when restricted to the subcategory of finite-dimensional vector spaces.

Functors may be thought of as general constructions. The existence of a natural equivalence between a pair of functors means that these constructions lead to essentially the same result.

An important mathematical concept whose relevance for theoretical physics has been recently recognized is that of a *fibre bundle*.[21] Fibre bundles generalize the notion of a Cartesian product; locally, they can always be represented as Cartesian products. One may see the need for such a generalization by considering the development of our ideas of space and time.

According to the Ancient Greeks' picture of the world, space-time† E was a Cartesian product of time T and space S[22]: to any event one could associate an instant of time t and a location in space s; both time and space were absolute.

In Newtonian physics, space-time E may be represented as a product $T \times S$ in many ways; none of these representations are natural in a sense, which can be easily related to the notation of natural transformations. Space is relative because there is no absolute method of ascertaining whether or not two non-simultaneous events happen at the same place. In other words, there is no natural horizontal slicing of E; there is only a vertical fibring corresponding to the projection $\pi: E \to T$ which associates to any event $p \in E$ the corresponding instant of time $t = \pi(p)$; or, time is absolute.

The last example provides us with the essential set-theoretic ingredients of a *bundle* A: it consists of two sets, say M and E, called respectively the *bundle space* (or the total space) and the *base space*, and of a surjective map $\pi: M \to E$, called the *projection*; shortly $A = (M, E, \pi)$. The sets M and E usually have some additional structure, such as that of a topological space or a differential manifold, and π is then assumed to be compatible with these structures (i.e., to be a morphism in the corresponding category). In most cases, the spaces $\pi^{-1}(p)$, $p \in E$, are all alike, i.e., isomorphic (in that category) to a space F, called the *typical fibre*; the set $\pi^{-1}(p)$ is then called the *fibre* over p. For any $p \in E$, there is usually more than one isomorphism of $\pi^{-1}(p)$ onto F (otherwise M would admit a natural representation as $E \times F$); if f and f' are two such isomorphisms, then $f' \circ f^{-1}$ is an automorphism of F; the set of all automorphisms of F which are of this form constitutes a group, called the *structure group* G of the bundle.

In physics, we are most often interested in differential bundles: M, E and F are differential manifolds, π is differentiable and for any $p \in E$ there exists a neighbourhood U of p and a differential isomorphism (diffeomorphism) $h: \pi^{-1}(U) \to U \times F$ such that $pr_1 \circ h = \pi$.

The class of all differential bundles forms a category $\mathscr{B}un$; its morphisms are pairs (f, a) of mappings such that $\pi_2 \circ f = a \circ \pi_1$ with (M_1, E_1, π_1) and (M_2, E_2, π_2) denoting the bundles. A product bundle is $(E \times F, E, \mathrm{pr}_1)$, where $\mathrm{pr}_1(p, q) = p$. A bundle $\mathscr{B}un$-isomorphic to a product bundle is called *trival.*

Among differential bundles especially important are *vector bundles* and *principal bundles*. Roughly speaking, a vector bundle is a differential bundle with a vector space playing the role of the typical fibre. For example, the tangent bundle $T(E)$ and the

† Strictly speaking, one should distinguish between the physical space-time and models used to describe it. A sentence like 'according to Einstein, space-time is a Riemannian manifold' should really read 'in Einstein's theory a Riemannian manifold is used as a model of space-time'. We shall adhere, however, to the convenient abuses of language which prevail in the physical literature.

cotangent bundle $T^*(E)$ of a manifold E are vector bundles over E. There is a covariant functor T from $\mathscr{Diff}$ into the category of vector bundles $\mathscr{VBun}$; $T(f)$ is the tangent mapping of the differentiable map f. A principal bundle has a Lie group as typical fibre and structure group at the same time; the group acts freely in the bundle and transitively on its fibres. For example, the bundle of frames $B(E)$ of a manifold E is a principal bundle with structure group $GL(n)$, where $n = \dim E$. There is a covariant functor B from $\mathscr{IDiff}$ into the category of principal bundles $\mathscr{PBun}$.

The functors $T, \tau: \mathscr{Aff} \to \mathscr{VBun}$, where T is the tangent functor and

$$\tau(A) = E \times V, \qquad A = (E, V, +)$$

are naturally equivalent to each other. The natural transformation

$$N(A): E \times V \to T(E) \tag{2}$$

associates to $(p, v) \in E \times V$ the vector $X \in T(E)$ tangent to the curve $t \mapsto tv + p$ at p.

We may now list a number of categories that are frequently used in physics. In most cases, it is possible to restrict the category by specifying the number of dimensions of its objects; this will not be done here because the dimensionality of space-time does not enter our elementary considerations. We shall only assume that all manifolds and vector spaces are finite dimensional.

I. The *Galilean category* $\mathscr{Gal}$ has Galilean spaces as objects. A Galilean space is an affine space $(E, V, +)$ endowed with a bilinear map

$$h: V^* \times V^* \to \mathbf{R}$$

which is (*a*) symmetric, (*b*) positive, and (*c*) of rank $n-1$, where $n = \dim V$. If $(E_1, V_1, +, h_1)$ and $(E_2, V_2, +, h_2)$ are two Galilean spaces, the affine morphism f is a Galilean morphism if $h_1 \circ (\tau f)^* = h_2$. A Galilean automorphism is called a Galilean transformation.

Let $\Sigma \subset V^*$ be the null space of h and $S \subset V$ the subspace of all vectors orthogonal to Σ. For any Galilean space $A = (E, V, +, h)$ the quotient space $T = E/S$ is called the absolute time of A. If

$$\pi: E \to T$$

is the canonical projection, then (E, T, π) is a fibre bundle with S as the typical fibre. The relation of absolute time to Galilean transformations is described by the following proposition: There is a covariant functor σ: $\mathscr{Gal} \to \mathscr{Bun}$ defined by

$$\sigma(A) = (E, T, \pi) \tag{3}$$

and

$$\sigma(f) = (f, a)$$

where $f: E_1 \to E_2$ is a Galilean morphism and $a: T_1 \to T_2$ is the unique map satisfying $\pi_2 \circ f = a \circ \pi_1$.

II. The *category of phase spaces*, $\mathscr{Phase}$, plays a role in classical mechanics. A phase space is a pair (M, β) consisting of an even-dimensional differential manifold M and a non-degenerate two-form field β on M. Morphisms of $\mathcal{P}hase$ are defined as differentiable maps carrying one two-form into another. Automorphisms of $\mathcal{P}hase$ are called canonical transformations. In classical mechanics, one makes frequent use of the existence of the following two functors:

$$\mathcal{I}\mathcal{D}iff \xrightarrow{T^*} \mathcal{P}hase \xrightarrow{\Pi} \mathcal{L}ie\,\mathcal{A}lg \tag{4}$$

T^* associates with a differential manifold E the phase space $(T^*(E), \beta_E)$, where $\beta_E = \mathrm{d}\alpha_E$ and α_E is the canonical form on $T^*(E)$; Π is the Poisson functor mapping (M, β) into the Lie algebra of differentiable functions on M, with a bracket defined by β; the images of morphisms under T^* and Π are defined in an obvious way.

III. The *classical theory of fields* has $\mathcal{V}\mathcal{B}un$ as the underlying category. Let σ denote a differentiable action of $GL(n)$ in R^m; $\sigma: GL(n) \to GL(m)$, $\sigma_{id} = id$ and $\sigma_a \circ \sigma_b = \sigma_{ab}$ for any $a, b \in GL(n)$. One defines the functor $\bar{\sigma}$: $\mathcal{I}\mathcal{D}iff_n \to \mathcal{V}\mathcal{B}un$ by introducing in the set

$$\bar{\sigma}(E) = (B(E) \times \mathbf{R}^m)/GL(n)$$

the structure of a vector bundle over the n-dimensional differential manifold E; the action of $GL(n)$ in $B(E) \times \mathbf{R}^m$ is defined by $(e, q) \mapsto (ea, \sigma_{a^{-1}}(q))$ for any $a \in GL(n)$, $e \in B(E)$ and $q \in \mathbf{R}^m$. A cross-section of $\bar{\sigma}(E)$ is called a field of quantities of type σ. For example, if σ is the obvious representation of $GL(n)$ in the space of k-forms, $\mathbf{R}^m = \Lambda^k \mathbf{R}^{n*}$, then $\bar{\sigma}(E) = \Lambda^{k*}(E)$ is the bundle of k-forms over E.

Let $\mathcal{V}\mathcal{B}un_n$ denote the category of vector bundles over n-dimensional differential manifolds. If (M_i, E_i, π_i), $i = 1, 2$, are two such bundles, then $f: M_1 \to M_2$ is a morphism of $\mathcal{V}\mathcal{B}un_n$ if it is differentiable, admits a differentiable isomorphism $a: E_1 \to E_2$ such that $\pi_2 \circ f = a \circ \pi_1$ and is linear on each fibre. For any integer k one defines the kth jet extension functor[23]

$$J^k : \mathcal{V}\mathcal{B}un_n \to \mathcal{V}\mathcal{B}un_n$$

which plays a basic role in the theory of partial differential equations and in particular for classical fields. A $\mathcal{V}\mathcal{B}un_E$-morphism

$$\lambda: J^k \circ \bar{\sigma}(E) \to \Lambda^{n*}(E)$$

is called a Lagrangian for the field of quantities of type σ. The fundamental relations between Lagrangians, Euler-Lagrange equations, invariant transformations and conservation laws may be given a natural and simple formulation in this framework.

IV. The underlying category of *quantum physics* is that of complex Hilbert spaces, $\mathcal{H}ilb$. Its morphisms are unitary mappings. Let (E, μ) be a manifold E with a differentiable measure μ; one associates to it the Hilbert space $L^2(E, \mu)$ of square integrable complex functions, with a scalar product defined by

$$(f \mid g) = \int_E \bar{f} g \mu .$$

The class of all manifolds with differentiable measures constitutes a category $\mathscr{M}$; its morphisms are measure-preserving diffeomorphisms. There is a covariant functor $L^2 : \mathscr{M} \to \mathscr{H}ilb$ which assigns to an $\mathscr{M}$-morphism h the unitary transformation $L^2(h)$ defined by $L^2(h)(f) = f \circ h^{-1}$. Moreover, to a vector field X on E which preserves μ,

$$\mathscr{L}_X \mu = 0.$$

the functor L^2 associates the antihermitean operator $\mathscr{L}_X$ on $L^2(E, \mu)$, i.e., the Lie derivative with respect to X. In other words, L^2 gives rise to a functor $\mathscr{M} \to \mathscr{L}ie\,\mathscr{A}lg$. The study of its relation to the functor $\Pi \circ T^*$ defined by (4), is known as the problem of quantizing a mechanical system.

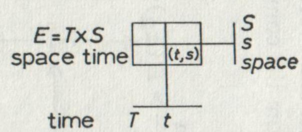
The principle of relativity as formulated in the preceding section, implies that there are no privileged inertial systems and this, in turn, may be interpreted to mean that Galilean space-time is not a Cartesian product of space and time (cf. Fig. 1). The principle of general invariance has a similar consequence. In the theory of special relativity, space-time is a flat (affine) space, i.e., a Riemannian space with an integrable linear connection. The existence of distant parallelism in this case implies that $B(E)$, the bundle of frames, is a Cartesian product. This is no longer so in general relativity where the result of transferring a vector from one point to another by parallel transport along a curve depends on that curve. One is tempted to say that the bundle of frames of space-time in general relativity is not trivial. However, according to the precise definition given above, $B(E)$ is trivial unless E has a non-Euclidean topology. Any global coordinate system on E induces an isomorphism of $B(E)$ onto $E \times GL(n)$. Nevertheless, the intuitive property of '$B(E)$ not being a product' may be given a precise formulation in terms of categories.

On the category $\mathscr{I}\mathscr{A}ff$, which is appropriate for both Galilean physics and special relativity, one can define two functors to $\mathscr{P}\mathscr{B}un$. One of them is the functor B associating to $E \in \mathscr{I}\mathscr{D}iff$, the bundle of frames $B(E)$, the second, C, is a functor of constructing the product bundle

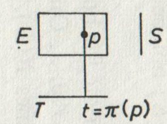
$$C(A) = E \times B(V),$$

where $A = (E, V, +)$ and $B(V)$ denotes the set of all vector frames of V. These functors are naturally equivalent, as may be seen from the commutative diagram

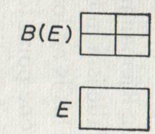
$$\begin{tikzcd}
B(E_1) \arrow[r, "B(f)"] \arrow[dd, bend right=60, "BN(A_1)"'] & B(E_2) \arrow[dd, bend left=60, "BN(A_2)"] \\
A_1 \arrow[u, |->] \arrow[d, |->] \arrow[r] & A_2 \arrow[u, |->] \arrow[d, |->] \\
E_1 \times B(V_1) \arrow[r, "f \times B(\tau f)"'] & E_2 \times B(V_2)
\end{tikzcd}$$



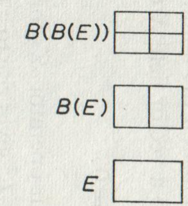
Aristotelian space-time E is the Cartesian product of time T and space S. Both space and time are absolute. This structure is assumed also in prerelativistic electrodynamics.



Galilean space-time E is a fibre bundle over the base T (time).



The Minkowski space-time does not even have a natural fibring. But the bundle of frames $B(E)$ is a product bundle.



In *GRT*, the bundle of frames is not a product but admits teleparallelism which turns $B(B(E))$ into a product.

Fig. 1. Historical development of ideas on the structure of space and time.

where f is any affine isomorphism and $BN(A)$ is the $\mathscr{PB}un$-isomorphism induced from the $\mathscr{VB}un$-isomorphism $N(A)$ described in Equation (2). Nothing analogous exists for the larger category $\mathscr{ID}iff$; the bundles $B(E)$ and $E\times B(\mathbf{R}^n)$ may be isomorphic but there is no natural equivalence of the corresponding functors.

A similar analysis may be applied to show that, in the Galilean category, the functor σ occurring in Equation (3) is not naturally equivalent to the 'product functor' $\mathscr{G}al\to\mathscr{B}un$, associating with $A\in\mathscr{G}al$ the product bundle $(T\times S, T, pr_1)$.

Keeping in mind that all statements about spaces being or not being products should be understood as referring to the natural equivalence of appropriate functors, we may now *compare the meaning of the two principles of relativity*:

(*a*) the special principle implies that space-time E is not a product;

(*b*) the general principle implies that the bundle of frames $B(E)$ is not a product. These are analogous statements but they refer to different spaces and no wonder that this has led to numerous controversies in the past.

In the theory of general relativity, one can take as the underlying category that of differential manifolds with linear connections. A linear connection on E induces a privileged field of linear bases on $B(E)$ and thus turns $B(B(E))$ into a product.[25] One may speculate as to the existence of a theory of space-time in which $B(B(E))$ is not a product.† Even if it should turn out that this generalization is physically uninteresting, it is clear that fibre bundles provide us with a deep insight into the structure of space-time and the nature of its theories.

For a long time, it has been recognized that '*gauge-invariant theories*', such as electrodynamics, are conveniently described in terms of principal bundles with connections. If P is a principal bundle over the base E, with structure group G and a connection form ω, the group G may be interpreted as the group of gauge transformations of the first kind. For any cross-section $\phi: E\to P$, the form on E

$$A=\phi^*\omega$$

is the 'potential', whereas the two-form

$$F=\phi^*\Omega$$

$$\Omega=\mathrm{d}\omega+\omega\wedge\omega$$

is the 'field' arising from gauge invariance. A change of the cross-section of P induces a change in A interpreted as a gauge transformation of the second kind. For $G=\mathbf{SO}(2)$ and $\mathbf{SO}(3)$ one obtains, in this way, the potentials and the fields of the Maxwell and Yang-Mills theories, respectively.[26]

An infinitesimal connection in P may be used to define covariant differentiation in vector bundles associated to P. In particular, if one constructs a complex vector

† D. D. Ivanenko suggested that the construction of such a theory should be referred to as the second relativization. It has been pointed out by F. A. E. Pirani that a theory of space-time based on conformal geometry is of this type.

bundle associated to the electromagnetic bundle by the homomorphism

$$\sigma(e^{i\alpha}) = e^{in\alpha}, \qquad \alpha \in \mathbf{R}, \quad n \in \mathbf{N},$$

he obtains for the covariant derivative, expressed in terms of local coordinates,

$$\nabla_k = \partial_k - inA_k, \quad (k = 1, 2, 3, 4),$$

and this may be used to justify the known form of the minimal electromagnetic coupling.

It is clear that the underlying category for the theory of fields connected with gauge invariance with respect to a Lie group G is that of principal bundles over space-time, with G as the structure group. A choice of a particular physical situation implies the choice of an object in the category, together with an infinitesimal connection; choosing a gauge is equivalent to picking up a cross-section of the bundle. All gauges are on the same footing (principle of relativity of gauges) because a bundle P over E is not naturally isomorphic to $E \times G$, in the sense explained above.

It is sometimes asserted that the general theory of relativity may also be obtained in this way, by taking G to be the Lorentz group or the Poincaré group.[27] This is not quite the case: the general-relativistic principal bundle has a structure richer than that of a bundle with the Lorentz group $\mathbf{O}(1, 3)$ as the typical fibre. In other words, the underlying category of general relativity is essentially narrower than that of principal bundles with $\mathbf{O}(1, 3)$ as the structure group. This is due to the following theorem: a principal fibre bundle P over an n-dimensional manifold E and with structure group $GL(n)$ is $\mathscr{PBun}$-isomorphic to $B(E)$ if and only if there exists an $\mathbf{R}^n$-valued one-form θ on P, such that

$$\theta(X) = 0 \Leftrightarrow T(\pi)X = 0,$$

$$\psi_a^* \theta = a^{-1}\theta,$$

where $\pi: P \to E$ is the projection and $\psi_a: P \to P$ denotes the action of $a \in GL(n)$ in P. The form θ is often called the 'soldering form' of P; the bundle $B(E)$ is 'soldered' to E rather than being loosely connected to E, as general principal bundles are. The covariant exterior differential of θ is the torsion form of the connection. Note also, that, for any manifold E, one can introduce the product bundle $E \times GL(n)$. In general, not only there is no natural isomorphism of $B(E)$ on $E \times GL(n)$ but no global isomorphism whatsoever (e.g., if E is a two-sphere).

A disadvantage of the gauge approach to electrodynamics is that it does not provide a natural method of deriving the other half of Maxwell equations (i.e., other than $dF=0$). The full set of Maxwell equations is known to follow from a simple action principle in the Kaluza–Klein theory, or one of its modifications.[28] It is interesting to know that, in fact, there is a definite isomorphism between the theory based on an infinitesimal connection and the Kaluza–Klein five-dimensional theory. This isomorphism may be extended to a large class of theories with gauge invariant fields. In other words, for any such theory it is possible to construct a multidimensional, Riemannian space which bears the same relation to that theory as the Kaluza–Klein space to electrodynamics.[29]

Let G be a Lie group, possessing an invariant metric h, i.e., a symmetric non-degenerate covariant tensor field of second order, defined on G and invariant with respect to both left and right translations. For example, if G is semi-simple, then one can define h by $h_e(A, B) = Tr(Ad_A \circ Ad_B)$ where $Ad_A(C) = [A, C]$, A, B, C belong to the Lie algebra of G and e is the unit of G. An abelian group, such as $\mathbf{SO}(2)$, also has an invariant metric. Given a principal fibre bundle P with structure group G, over a base manifold E (space-time) with a Riemannian metric g, one can define a Riemannian metric γ on P as follows. Let X be a vector tangent to P at r and write $\gamma(X)$, $h(A)$, etc., instead of $\gamma(X, X)$, $h(A, A)$, etc. We put

$$\gamma_r(X) = g_{\pi(r)}(T(\pi)X) + h_e(\omega(X)).$$

It follows from the properties of h that γ is non-singular and invariant with respect to G,

$$\psi_a^* \gamma = \gamma \quad \text{for any} \quad a \in G.$$

In particular, if $G = \mathbf{SO}(2)$, its Lie algebra can be identified with $\mathbf{R}$ and h may be taken to be the Euclidean metric on $\mathbf{R}$ (possibly, with a numerical coefficient), then γ on P is the Riemannian metric of the five-dimensional Kaluza–Klein theory. It is also clear how one can construct a principal fibre bundle from the Kaluza–Klein space.

This construction, when applied to the theory of a general field arising from gauge invariance, leads to the following possibility. One can consider the action integral $\int_{\pi^{-1}(\Omega)} K$ where K is the Ricci form, corresponding to the metric γ, and $\Omega \subset E$. By varying this action, with due care not to spoil the invariance of γ with respect to G, one obtains a set of field equations, analogous to the Einstein–Maxwell set that one gets in the Kaluza–Klein theory.

4. THE EINSTEIN–CARTAN THEORY

In 1922 Elie Cartan[30] proposed a slight modification of Einstein's theory of gravitation. According to Cartan, space-time corresponding to a distribution of matter with an intrinsic angular momentum should be represented by a curved manifold with *torsion*, the latter being *related to the density of spin*. This idea may be made plausible by the following considerations.

In the theory of special relativity, the group of inhomogeneous Lorentz transformations (the Poincaré group) plays a fundamental role in the description of elementary physical phenomena. In Cartesian coordinates (x^i), an infinitesimal Poincaré transformation is of the form

$$\delta x^i = \lambda^i{}_j x^j + \mu^i \quad (i, j = 1, 2, 3, 4) \tag{5a}$$

where

$$\lambda_{ij} + \lambda_j^i = 0. \tag{6a}$$

The Lie algebra of the Poincaré group has two basic invariants, interpreted physically as *mass* and *spin*. In Einstein's theory of general relativity, mass directly influences curvature but spin has no similar dynamical effect. On the other hand, curvature and torsion are related, respectively, to the groups of homogeneous transformations and of translations in the tangent spaces of a manifold endowed with a linear connection. Indeed, let (θ^i) be a field of co-frames, i.e. a set of four fields of one-forms which are linearly independent at each point of the manifold. If $(\omega^i{}_j)$ are the one-forms of the linear connection with respect to (θ^i), then the curvature and torsion two-forms are, respectively,

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = \tfrac{1}{2}\theta^k \wedge R^i{}_{jk} = \tfrac{1}{2}R^i{}_{jkl}\theta^k \wedge \theta^l\,,$$

and

$$\Theta^i = \mathrm{d}\theta^i + \omega^i{}_j \wedge \theta^j = \tfrac{1}{2}\theta^j \wedge Q^i{}_j = \tfrac{1}{2}Q^i{}_{jk}\theta^j \wedge \theta^k\,.$$

Denoting by D the exterior covariant derivative[25], one can define a radius-vector as a field (x^i) such that

$$Dx^i = \theta^i\,. \tag{7}$$

In a general curved space Equation (7) has no solutions but it can always be integrated along a curve. When this is done for a loop, one finds that the vector (x^i) does not return to its initial value. For an infinitesimal closed curve, the change in the radius-vector is

$$\delta x^i = (\Omega^i{}_j x^j - \Theta^i) \times \text{surface element}\,. \tag{5b}$$

If the linear connection is metric,

$$Dg_{ij} = 0\,, \tag{8}$$

then

$$\Omega_{ij} + \Omega_{ji} = 0\,. \tag{6b}$$

In other words, the curvature and torsion induce, respectively, a Lorentz transformation and a translation of the radius-vector field constructed along a closed curve in a space with a metric linear connection.

Cartan's basic idea has been developed by several authors.[31] The generalization due to Cartan constitutes a slight departure from Einsteins' theory: the field equations in empty space remain unchanged. In our opinion, the Einstein–Cartan theory is the simplest and the most natural modification of the original, Einstein's theory of gravitation. This modification deserves to be analyzed in detail, in precedence over the theories requiring an additional scalar field to describe gravitational phenomena.

The desirability of such an analysis may be related to recent discoveries in astronomy. It is conceivable that torsion may produce observable effects inside those objects which, as the neutron stars, have built-in strong magnetic fields, possibly accompanied by a substantial average value of the density of spin. One is tempted to speculate that the intrinsic angular momentum may influence – or even prevent – the occurrence of singularities in gravitational collapse and cosmology. A recent result of W. Kopczyński[32] supports this idea.

For a body with given values of spin and mass, the dimensionless numbers characterizing the order of magnitude of the effects of torsion and of curvature are, respectively,

$$\text{spin}/(\text{radius})^2 \quad \text{and} \quad \text{mass}/\text{radius}.$$

(We use a system of units in which the gravitational constant and the velocity of light are equal to 1.) For an electron, the ratio of these two (very small) numbers is of the order of $1/\alpha \approx 137$; the influence of spin on geometry is larger than that of mass. This is no longer so for matter in bulk because mass is essentially additive whereas in most circumstances spins cancel out one another.

The *field equations* of the Einstein–Cartan theory can be derived from a variational principle,

$$\delta \int (K + L) = 0,$$

where L is the Lagrangian (four-form) of matter and K is the Ricci four-form,

$$16\pi K = \eta_k{}^l \wedge \Omega^k{}_l$$

and

$$\eta_{ij} = \tfrac{1}{2}\theta^k \wedge \eta_{ijk} = \tfrac{1}{2}\eta_{ijkl}\theta^k \wedge \theta^l,$$

$$\eta_{ijkl} = \eta_{[ijkl]}, \quad \eta_{1234} = |\det g_{mn}|^{1/2}.$$

If the sources of the gravitational field are described by a tensor or spinor field (φ_A), then, by varying with respect to θ^i, $\omega^k{}_l$ and φ_A, and assuming Equation (8), one arrives at the system of equations

$$\tfrac{1}{2}\eta_{ijk} \wedge \Omega^{jk} = -\,8\pi t_i, \tag{9}$$

$$\eta_{ijk} \wedge \Theta^k = 8\pi s_{ij}, \tag{10}$$

$$L_A = 0, \tag{11}$$

where t_i and s_{ij} are the densities (three-forms) of energy-momentum and of spin, respectively. In the absence of spin, the energy-momentum density is symmetric

$$\theta^k \wedge t^l = \theta^l \wedge t^k$$

and Equation (9) goes over into the Einstein equation. In the general case, there is a symmetric energy-momentum tensor

$$T^{ij} = \theta^j \wedge t^i - \tfrac{1}{2}Ds^{ij}.$$

In the approximation of special relativity there is a radius vector (x^i) subject to Equation (7) and the conservation law of energy-momentum, $Dt_i = 0$, together with the symmetry of T^{ij}, implies that the total angular momentum is conserved:

$$\text{if} \quad L_A = 0, \quad \text{then} \quad D(x^i t^j - x^j t^i + s^{ij}) = 0.$$

Similarly as in Einstein's theory, the *equations of motion* can be deduced, in simple cases, directly from the field Equations (9) and (10), without using Equation (11). The Bianchi identities for a curved space with torsion, applied to Equations (9) and

(10) yield the relations

$$Dt_j = Q^k{}_j \wedge t_k - \tfrac{1}{2}R^{kl}{}_j \wedge s_{kl}, \tag{12}$$

$$Ds_{kl} = \theta_l \wedge t_k - \theta_k \wedge t_l. \tag{13}$$

In the absence of spin, Equation (12) reduces to the usual covariant conservation law $Dt_j = 0$.

To derive the equations of motion of a spinning fluid or dust, it is convenient to introduce a 'particle derivative' defined as follows.[33] Let $u = v^i \eta_i$ be the three-form dual with respect to the velocity vector field (v^i), $\eta_i = \frac{1}{3}\theta^j \wedge \eta_{ij}$. The particle derivative of a tensor field (φ_A) with respect to (v^i) is given by the formula

$$\dot{\varphi}_A \eta = D(\varphi_A u),$$

where $\eta = \frac{1}{4}\theta^i \wedge \eta_i$ is the volume element in space-time. A *spinning dust* may be defined as a continuous medium characterized by its velocity (v^i), the density of energy and momentum (P_i), and the density of spin (S_{ij}). The three-forms of energy-momentum and of spin are

$$t_i = P_i u \quad \text{and} \quad s_{ij} = S_{ij} u,$$

respectively. From Equation (13) there follows the relation

$$P^i = \varrho v^i + v_k \dot{S}^{ki}, \tag{14}$$

where $\varrho = g_{ij} P^i v^j$ and $\dot{S}^{ij}$ is the particle derivative of S^{ij} with respect to (v^i). Equation (13) is equivalent to the system consisting of Equation (14) and the equation of *motion of spin*,

$$\dot{S}^{ij} = v^i v_k \dot{S}^{kj} - v^j v_k \dot{S}^{ki}.$$

The modified conservation law given by Equation (12) gives rise to the equation of *translatory motion*

$$\dot{P}_i = Q^k_{ij} v^j P_k + \tfrac{1}{2}R^{kl}{}_{ij} v^j S_{kl} \tag{15}$$

which is a generalization, to the Riemann–Cartan space of an equation derived by Mathisson[34] and Papapetrou[35] for point particles with an intrinsic angular momentum. If the dust has no spin, $S_{kl} = 0$, then there is no torsion, and the equations of motion are simply

$$\dot{\varrho} = 0 \quad \text{and} \quad \dot{P}_i = 0.$$

The Einstein–Cartan theory gives rise to a number of interesting possiblitities. According to F. Hehl[36] the new theory may contribute to an explanation of weak interactions. A more conservative attitude is to look for new macroscopic effects in regions with strong magnetic fields. M. A. Melvin[37] has suggested that torsion may play a significant role during the early stage of the development of the Universe.

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