28. IV

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THEORETICAL PHYSICS

On the Einstein—Cartan Equations. IV

by

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Summary. It is shown that with every one-parameter group of symmetries of a Riemann—Cartan space-time there is associated a conservation law $dj=0$ for the current of energy, momentum and/or spin.

1. Symmetries of a Riemann—Cartan space. Let (X, g, ω) be a Riemann—Cartan space, i.e., a differential manifold X with a metric tensor g and a linear connection ω compatible with g. The *transposed connection* $\tilde{\omega}$ is defined in terms of its components $\tilde{\omega}^i_j$ relative to a field of frames (θ^i),

$$\tilde{\omega}^i_j=\omega^i_j+Q^i_j,$$

where $Q^i_j=Q^i_{jk}\,\theta^k$ and Q^i_{jk} is the tensor of torsion of the connection ω. The transposed connection is metric if and only if $Q_{ijk}=Q_{[ijk]}$. If ω^i_j and $\tilde{\omega}^i_j$ are written as $\Gamma^i_{jk}\,\theta^k$ and $\tilde{\Gamma}^i_{jk}\,\theta^k$, respectively, then, for a holonomic frame (θ^i),

$$\tilde{\Gamma}^i_{jk}=\Gamma^i_{kj}.$$

The covariant derivative of a vector field v, relative to $\tilde{\omega}$, is

$$\theta^j\,\tilde{\nabla}_j\,v^i=\tilde{D}v^i=Dv^i+v\,\lrcorner\,\Theta^i.$$

Similarly, if (t_i) is a covector-valued p-form, then

$$\tilde{D}t_i=Dt_i-Q^j_i\wedge t_j.$$

A *symmetry* (automorphism) of (X, g, ω) is a diffeomorphism of X which preserves g and ω. Consider a one-parameter group of transformations of X generated by the vector field v. A necessary and sufficient condition for the transformations to be symmetries of (X, g, ω) is that the Lie derivatives of g and ω with respect to v vanish [1]:

$$\tilde{\nabla}^i\,v^j+\tilde{\nabla}^j\,v^i=0, \tag{1}$$

$$D\tilde{\nabla}_j\,v^i+v\,\lrcorner\,\Omega^i_j=0. \tag{2}$$

In a Riemannian space, the connections ω and $\tilde{\omega}$ coincide and (2) is a consequence of the Killing equation (1).

2. Conservation laws in the Einstein—Cartan theory. Similarly as in Einstein's theory of general relativity, symmetries of a Riemann—Cartan space-time give rise to conservation laws in the form of an "ordinary divergence", $dj=0$. The conservation theorem may be obtained from the Einstein—Cartan field equations,

$$e_i=-8\pi t_i, \qquad c_{kl}=-8\pi s_{kl},$$

by making use of the differential identities [2] which imply

$$\tilde{D}t_i=\tfrac{1}{2}R_i^{kl}\wedge s_{kl}, \tag{3}$$

$$Ds_{kl}=\theta_k\wedge t_l-\theta_l\wedge t_k. \tag{4}$$

The exterior derivative of the three-form

$$j=v^i t_i+\tfrac{1}{2}\tilde{\nabla}^l v^k s_{kl}, \tag{5}$$

may be evaluated as follows:

$$dj=\tilde{D}(v^i t_i)+D(\tfrac{1}{2}\tilde{\nabla}^l v^k s_{kl})=v^i(\tilde{D}t_i-\tfrac{1}{2}R_i^{kl}\wedge s_{kl})+\tfrac{1}{2}\tilde{\nabla}^l v^k(Ds_{kl}-\theta_k\wedge t_l+\theta_l\wedge t_k)+$$
$$+\tfrac{1}{2}(\tilde{\nabla}^k v^l+\tilde{\nabla}^l v^k)\,\theta_k\wedge t_l+\tfrac{1}{2}(D\tilde{\nabla}^l v^k+v\lrcorner\,\Omega^{kl})\wedge s_{kl}.$$

By comparing the last two lines with Eqs. (1)—(4) one arrives at the

THEOREM. *If v generates a one-parameter group of symmetries of a Riemann—Cartan space-time, then there holds the conservation law*

$$dj=0, \tag{6}$$

for the 'current' j defined by (5), *where* (t_i) *and* (s_{kl}) *are the densities of energy-momentum and of spin, respectively.*

The conservation law (6) is meaningful in the limiting case of special relativity. If v generates a translation, then $\nabla^k v^l=0$ and j becomes the projection onto v of the energy-momentum density. If $v^k=\lambda_l^k x^l$, where $\lambda_{kl}=\lambda_{[kl]}$ is a constant matrix and (x^k) is a radius-vector, then $j=\lambda_{kl}j^{kl}$, where $j^{kl}=x^k t^l-x^l t^k+s^{kl}$ is the tensor density of total angular momentum.

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А. Траутман, **Об уравнениях Эйнштейна — Картана, IV часть**

Содержание. В настоящей работе показано, что каждой однопараметрической группе симметрии пространства—времени Риманна—Картана соответствует закон сохранения $dj=0$ для тока энергии, импульса или спина.