

On the Einstein—Cartan Equations. III

by

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Summary. An analogue of the Mathisson—Papapetrou equation of motion of a spinning particle is derived from the field equations of the Einstein—Cartan theory of gravity.

1. The particle derivative. Consider the motion of a fluid described in the Newtonian theory by the velocity vector field \mathbf{v} and the density of mass ρ . Let $V(t)$ denote the volume occupied by a given, and the same for any t , set of particles of the fluid. The mass contained in $V(t)$,

$$m(t) = \int_{V(t)} \rho dx \wedge dy \wedge dz,$$

changes in time according to

$$\frac{dm}{dt} = \int_{V(t)} \dot{\rho} dx \wedge dy \wedge dz,$$

where

$$\dot{\rho} = \partial\rho/\partial t + \operatorname{div}(\rho \mathbf{v}).$$

The mass is conserved if and only if $\dot{\rho} = 0$.

Let X be a Riemann—Cartan space-time [1] and let $v = (v^i)$ be a velocity field, i.e., a smooth vector field on X , satisfying $g_{ij} v^i v^j = 1$. The 3-form on X , dual with respect to $g_{ij} v^i \theta^j$ is

$$u = v \lrcorner \eta,$$

where η is the volume element of X . For any tensor field (φ_A) on X , we define its *particle derivative* $(\dot{\varphi}_A)$ with respect to v :

$$\dot{\varphi}_A \eta = D(\varphi_A u).$$

An equivalent definition is

$$\dot{\varphi}_A = v^i \nabla_i \varphi_A + \varphi_A \operatorname{div} v,$$

where $\operatorname{div} v$ is the divergence of v relative to η [2]. For a scalar field ρ , the condition $\dot{\rho} = 0$ is equivalent to the conservation law $d(\rho u) = 0$. The particle derivative is a convenient concept for the discussion of the relation between continuum mechanics and the description of point-like particles.

This note is a continuation of two previous papers [3]. Equations appearing in Part II are referred to in the style (II.*n*).

2. The generalized Mathisson-Papapetrou equation. A spinning dust is a continuous medium characterized by its velocity v , the density of energy and momentum (P_i), and the density of spin (S_{ij}). The 3-forms of energy-momentum and of spin are

$$t_i = P_i u \quad \text{and} \quad s_{ij} = S_{ij} u,$$

respectively. From Eq. (II.6) there follows the relation

$$(1) \quad P^i = \rho v^i + v_k \dot{S}^{ki},$$

where

$$\rho = g_{ij} P^i v^j$$

and \dot{S}^{ij} is the particle derivative of S^{ij} with respect to v . Eq. (II.6) is equivalent to the system consisting of Eq. (1) and the equation of *motion of spin*,

$$(2) \quad \dot{S}^{ij} = v^i v_k \dot{S}^{kj} - v^j v_k \dot{S}^{ki}.$$

The other differential identity, Eqs. (II.3) or (II.7), gives rise to the equation of *translatory motion*,

$$(3) \quad \dot{P}_i = Q_{ij}^k v^j P_k + \frac{1}{2} R_{ij}^{kl} v^j S_{kl},$$

which is a generalization, to the Riemann—Cartan space, of an equation derived by Mathisson [4] and Papapetrou [5, 6] for point particles with an intrinsic angular momentum. If the dust has no spin, $S_{kl} = 0$, then there is no torsion, and the equations of motion are simply

$$\dot{\rho} = 0 \quad \text{and} \quad \dot{P}_i = 0.$$

They imply that the field v is geodesic, $Dv^i \wedge u = 0$.

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А. Траутман, Об уравнениях Эйнштейна—Картана. III. часть

Содержание. На основании уравнений поля теории Эйнштейна—Картана в настоящей работе выведено обобщенное уравнение Матиссона—Папепетру, описывающее движение частиц со спином.