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THEORETICAL PHYSICS

On the Einstein—Cartan Equations. II.

by

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Summary. It is shown that by covariant exterior differentiation of the equations of the generalized theory of gravity one arrives at a system of relations which hold as an algebraic consequence of the field equations themselves. The Dirac equation in a Riemann-Cartan space-time is written in a manner adapted to the calculus of exterior forms.

1. The differential identities. The curvature and torsion forms of any affinely connected space satisfy the Bianchi identities*)

$$D\Omega^k_l=0 \quad \text{and} \quad D\Theta^k=\Omega^k_l\wedge\theta^l.$$

These identities may be used to evaluate the covariant exterior derivatives of the 3-forms e_j and c_{kl} appearing in the Einstein—Cartan equations of the generalized, metric theory of gravitation. It follows directly from (I.11) and (I.12) that

(1) $$De_j=\tfrac{1}{2}\eta_{jklm}\,\Theta^m\wedge\Omega^{kl},$$

(2) $$Dc_{kl}=\eta_{lj}\wedge\Omega^j_k-\eta_{kj}\wedge\Omega^j_l.$$

Eq. (1) is the generalization, to the Riemann-Cartan space, of the 'contracted Bianchi identity' $De_j=0$ which plays an important role in Einstein's theory of gravitation. The right-hand sides of Eqs (1) and (2) may be rearranged to give the formulae

(3) $$De_j=Q^k_j\wedge e_k+\tfrac{1}{2}R^{kl}{}_j\wedge c_{kl},$$

(4) $$Dc_{kl}=\theta_k\wedge e_l-\theta_l\wedge e_k,$$

where

$$\tfrac{1}{2}\theta^j\wedge Q^k_j=\Theta^k \quad \text{and} \quad \tfrac{1}{2}\theta^j\wedge R^{kl}{}_j=\Omega^{kl}.$$

Let σ be a representation of the Lorentz group in $\mathbf{R}^N$ and let (σ^{Bl}_{Ak}) be the matrix corresponding to the derived homomorphism of Lie algebras. The matrix $\sigma_{kl}=$

*) In this note, similar assumptions are made, and the same notation is used, as in Part I [1]. Equations appearing there are referred to in the style (I.n). The frames (θ^k) are assumed to be orthonormal so that $g_{kl}=0$ for $k\neq l$ and $g_{11}=g_{22}=g_{33}=-g_{44}=-1$.

$=(\sigma^B_{Akl})$, $\sigma^B_{Akl}=g_{jl}\,\sigma^{Bj}_{Ak}$, is skew, $\sigma_{kl}+\sigma_{lk}=0$. In the metric theory, the forms $\omega^{kl}=$ $=g^{jl}\,\omega^k_j$ of the connection, referred to a field (θ^k) of orthonormal frames, are also skew in the pair (k, l). The covariant derivative of a tensor-valued 0-form of type σ, $\varphi=(\varphi_A)$, is given by

$$D\varphi=d\varphi+\omega^{kl}\,\sigma_{kl}\,\varphi=\varphi_j\,\theta^j.$$

The indices $A, B=1, \ldots, N$ have been suppressed in this formula, which should be interpreted similarly as in spinor calculus.

Let the lagrangian form corresponding to φ be $L=\eta\Lambda\,(\varphi, \varphi_j)$, where Λ is a function on $\mathbf{R}^N\times\mathcal{L}\,(\mathbf{R}^4, \mathbf{R}^N)$. In this formulation, where everything is referred to orthonormal frames, there is no possibility for considering changes of g_{ij}. A general variation of L is

$$\delta L=t_j\wedge\delta\theta^j+\tfrac{1}{2}\,s_{kl}\wedge\delta\omega^{kl}+L^A\,\delta\varphi_A+\text{an exact form},$$

where

$$t_j=\left(\frac{\partial\Lambda}{\partial\varphi_k}\,\varphi_j-\Lambda\delta^k_j\right)\eta_k,\qquad s_{kl}=-2\,\frac{\partial\Lambda}{\partial\varphi_j}\,\sigma_{kl}\,\varphi\eta_j\,, \tag{5}$$

and

$$L^A=\eta\,\frac{\partial\Lambda}{\partial\varphi_A}-D\left(\eta_j\,\frac{\partial\Lambda}{\partial\varphi_{Aj}}\right).$$

A mere Lorentz rotation of the frames leaves both L and Λ invariant. This invariance results in the equation

$$Ds_{kl}=\theta_k\wedge t_l-\theta_l\wedge t_k\qquad\text{mod } L^A=0 \tag{6}$$

proving the symmetry of the tensor T_{kl} which should now be considered as *defined* by Eq. (I.6). The covariant exterior derivative of t_j may be evaluated from Eq. (5)

$$Dt_j=Q^k_j\wedge t_k+\tfrac{1}{2}\,R^{kl}{}_j\wedge s_{kl}\qquad\text{mod } L^A=0. \tag{7}$$

A simple comparison of Eqs. (I.8) with (3), (4), (6) and (7) leads to

THEOREM. *The equations resulting by covariant exterior derivation of the Einstein-Cartan equations are algebraic consequences of the Einstein-Cartan equations themselves and of the equations of motion of matter.*

The implications of this result should be compared with, and distinguished from, the point of view of Cartan who required that Dt_j be zero ([2], pp. 21—23).

2. The Dirac equation in a space-time with torsion. To illustrate the general formalism developed in [1] and in the preceding section, we give here a brief account of the Dirac equation appropriate to a Riemann-Cartan space-time. A detailed and different treatment of this subject may be found elsewhere [3].

Let X be a four-dimensional differential manifold endowed with a metric tensor g of hyperbolic signature. A simple-minded, physicist's approach to Dirac spinors on X may be summarized as follows. The Dirac matrices $\gamma_k\in\mathcal{L}\,(\mathbf{C}^4)$ satisfy

$$\gamma_k\,\gamma_l+\gamma_l\,\gamma_k=2g_{kl}$$

and $\beta=\beta^+$ is a matrix such that

$$\gamma_k^+=\beta\gamma_k\beta^{-1},$$

where cross denotes hermitean conjugation. The six spin matrices

$$\sigma_{kl}=\tfrac{1}{8}(\gamma_k\gamma_l-\gamma_l\gamma_k)$$

satisfy

$$\sigma_{kl}\gamma_j-\gamma_j\sigma_{kl}=\tfrac{1}{2}(g_{jl}\gamma_k-g_{jk}\gamma_l), \tag{8}$$

$$\sigma_{kl}\gamma_j+\gamma_j\sigma_{kl}=\tfrac{1}{2}\eta_{jklm}\gamma_5\gamma^m, \tag{9}$$

where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$, $\gamma_k=g_{kl}\gamma^l$, $\eta_{1234}=1$, etc. Space-time is assumed to have a global field of orthonormal frames (θ^k). One considers only those fields of orthonormal frames (θ'^k) which differ 'little' from a given field (θ^k),

$$\theta'^k=\theta^k+\delta\theta^k=\theta^k-a_l^k\theta^l \quad\text{and}\quad a_{kl}+a_{lk}=0.$$

A spinor field on X is a law which associates to each of the fields of frames a map from X to $\mathbf{C}^4$ in such a way that if $\psi: X\to\mathbf{C}^4$ corresponds to (θ^k) and $\psi'=\psi+\delta\psi$ corresponds to (θ'^k), then

$$\delta\psi=-a^{kl}\sigma_{kl}\psi.$$

Similarly, for the contragredient spinor $\bar\psi=\psi^+\beta$,

$$\delta\bar\psi=\bar\psi\sigma_{kl}a^{kl}.$$

Let X be endowed, in addition to the metric tensor, with a metric affine connection described by the collection (ω_{kl}) of 1-forms. The covariant derivative $D\psi$ of a spinor field ψ is a spinor-valued 1-form

$$D\psi=d\psi+\omega^{kl}\sigma_{kl}\psi.$$

The corresponding formula for the derivative of the contragredient spinor is

$$D\bar\psi=d\bar\psi-\bar\psi\sigma_{kl}\omega^{kl}.$$

The covariant exterior derivative of $\iota=\gamma_k\eta^k$ (the dual of $\gamma=\gamma_k\theta^k$),

$$D\iota=\iota\wedge Q_k^k$$

depends on the trace of the torsion tensor. If $*D\psi$ denotes the dual of $D\psi$, then

$$-\iota\wedge D\psi=\gamma\wedge *D\psi=\eta\gamma^k\nabla_k\psi. \tag{10}$$

The lagrangian form corresponding to a Dirac particle of mass m is

$$L=\tfrac{1}{2}i(\bar\psi\iota\wedge D\psi+D\bar\psi\wedge\iota\psi)+m\eta\bar\psi\psi.$$

The Dirac equation obtained by varying the action integral with respect to $\bar\psi$ or ψ,

$$\iota\wedge D\psi-D(\iota\psi)=2im\eta\psi, \tag{11}$$

$$(D\bar\psi)\wedge\iota+D(\bar\psi\iota)=2im\eta\bar\psi, \tag{12}$$

implies the conservation law $dj=0$, where $j=\bar{\psi}\iota\psi$ is the current. By varying L with respect to θ^j and ω^{kl}, and making use of Eqs (8)—(12), one obtains

$$t_j=\tfrac{1}{2}\,i\,(\bar{\psi}\gamma_j\,{}^*D\psi-{}^*D\bar{\psi}\gamma_j\,\psi) \tag{13}$$

and

$$s_{kl}=i\bar{\psi}\,(\iota\sigma_{kl}+\sigma_{kl}\,\iota)\,\psi=-\tfrac{1}{2}\,i\bar{\psi}\gamma_5\,\gamma\psi\wedge\theta_k\wedge\theta_l\,. \tag{14}$$

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А. Траутман, **Об уравнениях Эйнштейна — Картана. II.**

Содержание. В настоящей работе показано, что при использовании ковариантного внешнего дифференцирования уравнений обобщенной теории гравитации, получаем систему соотношений, которая является алгебраическим следствием самых уравнений поля. Уравнение Дирака в пространстве — времени Римана-Картана записано при использовании формализма внешних форм.