On the Einstein—Cartan Equations. II.

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Summary. It is shown that by covariant exterior differentiation of the equations of the generalized theory of gravity one arrives at a system of relations which hold as an algebraic consequence of the field equations themselves. The Dirac equation in a Riemann-Cartan space-time is written in a manner adapted to the calculus of exterior forms.

1. The differential identities. The curvature and torsion forms of any affinely connected space satisfy the Bianchi identities*)

\[ D\Omega_k = 0 \quad \text{and} \quad D\Theta^k = \Omega_k^j \wedge \theta^j. \]

These identities may be used to evaluate the covariant exterior derivatives of the 3-forms \( e_j \) and \( c_{kl} \) appearing in the Einstein—Cartan equations of the generalized, metric theory of gravitation. It follows directly from (I.11) and (I.12) that

\[ \begin{align*}
D e_j &= \frac{1}{2} \eta_{jkm} \Theta^m \wedge \Omega^k, \\
D c_{kl} &= \eta_{ijkl} \Omega^l - \eta_{kij} \Omega^j.
\end{align*} \]

Eq. (1) is the generalization, to the Riemann-Cartan space, of the ‘contracted Bianchi identity’ \( D e_j = 0 \) which plays an important role in Einstein’s theory of gravitation. The right-hand sides of Eqs (1) and (2) may be rearranged to give the formulae

\[ \begin{align*}
D e_j &= Q_j^k \wedge e_k + \frac{1}{2} R^k_{ij} \wedge c_{kl}, \\
D c_{kl} &= \theta_k \wedge e_i - \theta_i \wedge e_k,
\end{align*} \]

where

\[ \frac{1}{2} \theta^j \wedge Q_j^k = \Theta^k \quad \text{and} \quad \frac{1}{2} \theta^j \wedge R^k_{ij} = \Omega^k. \]

Let \( \sigma \) be a representation of the Lorentz group in \( \mathbb{R}^N \) and let \( (\sigma_{ak}) \) be the matrix corresponding to the derived homomorphism of Lie algebras. The matrix \( \sigma_{kl} = \]

*) In this note, similar assumptions are made, and the same notation is used, as in Part I [1]. Equations appearing there are referred to in the style (I.n). The frames \( (\theta^i) \) are assumed to be orthonormal so that \( g_{ki} = 0 \) for \( k \neq i \) and \( g_{11} = g_{22} = g_{33} = -g_{44} = -1. \)
\(= (\sigma_{AB}^B, \sigma_{AB}^B = g_{ij} \sigma_{i}^{B} \), is skew, \(\sigma_{k}^{i} + \sigma_{l}^{i} = 0\). In the metric theory, the forms \(\omega^{kl} = g^{kl} \omega^{k}_{j}\) of the connection, referred to a field \((\theta^{k})\) of orthonormal frames, are also skew in the pair \((k, l)\). The covariant derivative of a tensor-valued 0-form of type \(\sigma, \phi = (\phi_{\lambda})\), is given by
\[
D_{\phi} = d_{\phi} + \omega^{kl} \sigma_{kl} \phi = \phi_{j} \theta^{j}.
\]
The indices \(A, B = 1, ..., N\) have been suppressed in this formula, which should be interpreted similarly as in spinor calculus.

Let the lagrangian form corresponding to \(\phi\) be \(L = L_{A} (\phi, \phi_{j})\), where \(L\) is a function on \(R^{N} \times \mathcal{L} (R^{4}, R^{N})\). In this formulation, where everything is referred to orthonormal frames, there is no possibility for considering changes of \(g_{ij}\). A general variation of \(L\) is
\[
\delta L = t_{j} \wedge \delta \theta^{j} + \frac{1}{2} s_{kl} \wedge \delta \omega^{kl} + L_{A} \delta \phi_{A} + \text{an exact form},
\]
where
\[
t_{j} = \left( \frac{\partial L}{\partial \phi_{k}} \phi_{j} - L_{A} \delta^{k}_{j} \right) \eta_{k}, \quad s_{kl} = -2 \left( \frac{\partial L}{\partial \phi_{j}} \right) \sigma_{kl} \phi \eta_{j},
\]
and
\[
L_{A} = \eta \left( \frac{\partial L}{\partial \phi_{A}} - D \left( \eta_{j} \frac{\partial L}{\partial \phi_{AJ}} \right) \right).
\]

A mere Lorentz rotation of the frames leaves both \(L\) and \(A\) invariant. This invariance results in the equation
\[
D_{s_{kl}} = \theta_{k} \wedge t_{l} - \theta_{l} \wedge t_{k} \quad \text{mod} \quad L_{A} = 0
\]
proving the symmetry of the tensor \(T_{kl}\) which should now be considered as defined by Eq. (I.6). The covariant exterior derivative of \(t_{j}\) may be evaluated from Eq. (5)
\[
D_{t_{j}} = Q_{j}^{l} \wedge t_{k} + \frac{1}{2} R^{kl}_{j} \wedge s_{kl} \quad \text{mod} \quad L_{A} = 0.
\]
A simple comparison of Eqs. (I.8) with (3), (4), (6) and (7) leads to

**THEOREM. The equations resulting by covariant exterior derivation of the Einstein-Cartan equations are algebraic consequences of the Einstein-Cartan equations themselves and of the equations of motion of matter.**

The implications of this result should be compared with, and distinguished from, the point of view of Cartan who required that \(Dt_{j}\) be zero ([2], pp. 21—23).

**2. The Dirac equation in a space-time with torsion.** To illustrate the general formalism developed in [1] and in the preceding section, we give here a brief account of the Dirac equation appropriate to a Riemann-Cartan space-time. A detailed and different treatment of this subject may be found elsewhere [3].

Let \(X\) be a four-dimensional differential manifold endowed with a metric tensor \(g\) of hyperbolic signature. A simple-minded, physicist’s approach to Dirac spinors on \(X\) may be summarized as follows. The Dirac matrices \(\gamma_{k} \in \mathcal{L} (C^{4})\) satisfy
\[
\gamma_{k} \gamma_{l} + \gamma_{l} \gamma_{k} = 2g_{kl}
\]
and $\beta = \beta^+$ is a matrix such that
\[ \gamma_k^+ = \beta \gamma_k \beta^{-1}, \]
where cross denotes hermitean conjugation. The six spin matrices
\[ \sigma_{kl} = \frac{1}{6} (\gamma_k \gamma_l - \gamma_l \gamma_k) \]
satisfy
\begin{align*}
(8) & \quad \sigma_{kl} \gamma_j - \gamma_j \sigma_{kl} = \frac{1}{2} (g_{jl} \gamma_k - g_{jk} \gamma_l), \\
(9) & \quad \sigma_{kl} \gamma_j + \gamma_j \sigma_{kl} = \frac{1}{2} \eta_{jklm} \gamma_l \gamma^m,
\end{align*}
where $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$, $\gamma_k = g_{kl} \gamma^l$, $\eta_{1234} = 1$, etc. Space-time is assumed to have a global field of orthonormal frames ($\theta^k$). One considers only those fields of orthonormal frames ($\theta'^k$) which differ 'little' from a given field ($\theta^k$),
\[ \theta'^k = \theta^k + \delta \theta^k = \theta^k - a_k^t \theta^t \quad \text{and} \quad a_{kl} + a_{lk} = 0. \]
A spinor field on $X$ is a law which associates to each of the fields of frames a map from $X$ to $\mathbb{C}^4$ in such a way that if $\psi : X \to \mathbb{C}^4$ corresponds to ($\theta^k$) and $\psi' = \psi + \delta \psi$ corresponds to ($\theta'^k$), then
\[ \delta \psi = - a_k^l \sigma_{kl} \psi. \]
Similarly, for the contragredient spinor $\overline{\psi} = \psi^+ \beta$,
\[ \delta \overline{\psi} = \overline{\psi} \sigma_{kl} a_k^l. \]
Let $X$ be endowed, in addition to the metric tensor, with a metric affine connection described by the collection ($\omega_{kl}$) of 1-forms. The covariant derivative $D\psi$ of a spinor field $\psi$ is a spinor-valued 1-form
\[ D\psi = d\psi + \omega_{kl} \sigma_{kl} \psi. \]
The corresponding formula for the derivative of the contragredient spinor is
\[ D\overline{\psi} = d\overline{\psi} - \overline{\psi} \sigma_{kl} \omega^{kl}. \]
The covariant exterior derivative of $i = \gamma_k \eta^k$ (the dual of $\gamma = \gamma_k \theta^k$),
\[ D i = i \wedge Q^k \]
depends on the trace of the torsion tensor. If $^* D\psi$ denotes the dual of $D\psi$, then
\begin{align*}
(10) & \quad -i \wedge D\psi = \gamma \wedge ^* D\psi = \eta^k \nabla_k \psi.
\end{align*}
The lagrangian form corresponding to a Dirac particle of mass $m$ is
\[ L = \frac{1}{2} i \left( \overline{\psi} i \wedge D\psi + D\overline{\psi} \wedge i \psi \right) + m \eta \overline{\psi} \psi. \]
The Dirac equation obtained by varying the action integral with respect to $\overline{\psi}$ or $\psi$,
\begin{align*}
(11) & \quad i \wedge D\psi - D(i \psi) = 2im \eta \psi, \\
(12) & \quad (D\overline{\psi}) \wedge i + D(\overline{\psi} i) = 2im \eta \overline{\psi},
\end{align*}
implies the conservation law $dj=0$, where $j=\bar{\psi}i\psi$ is the current. By varying $L$ with respect to $\theta^j$ and $\omega^k$, and making use of Eqs (8)–(12), one obtains

$$t_j = \frac{1}{2} i (\bar{\psi}\gamma_j \ast D\psi - \ast D\bar{\psi}\gamma_j \psi)$$

and

$$s_{kl} = i\bar{\psi} (i\sigma_{kl} + \sigma_{kl} i) \psi = -\frac{1}{2} i\bar{\psi}\gamma_5 \gamma_\psi \wedge \theta_k \wedge \theta_l.$$

REFERENCES


А. Траутман, Об уравнениях Эйнштейна — Картана. II.

Содержание. В настоящей работе показано, что при использовании ковариантного внешнего дифференцирования уравнений обобщенной теории гравитации, получаем систему соотношений, которая является алгебраическим следствием самых уравнений поля. Уравнение Дира в пространстве — времени Римана-Картана записано при использовании формализма внешних форм.