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THEORETICAL PHYSICS

On the Einstein—Cartan Equations. I*)

by

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Summary. A new argument is presented to support the Cartan idea of modifying Einstein's theory of gravitation by relating torsion to intrinsic angular momentum. It is shown that the Einstein equation may be written in two equivalent forms, with either the symmetric or the canonical tensor density of energy and momentum as the source. The Cartan equation determines the linear connection only up to projective transformations; this arbitrariness may be removed by requiring that the connection be metric.

1. Introduction. In 1922 Élie Cartan [1] suggested a simple generalization of Einstein's theory of gravitation. He proposed to consider, as a model of space-time, a four-dimensional differential manifold with a metric tensor and a linear connection compatible with the metric but not symmetric, in general. According to Cartan, the torsion tensor of the connection should be related to the density of intrinsic angular momentum [2]. Independently of Cartan, similar ideas were put forward by several authors (for example, see [3—5]; the last paper contains other relevant references).

The following is a heuristic argument to support the Cartan proposal: by the holonomy theorems, curvature and torsion are related, respectively, to the groups of homogeneous transformations and of translations in the tangent spaces of a manifold. In the approximation of special relativity, the group of inhomogeneous Lorentz transformations and its invariants (mass and spin) play a fundamental role in the description of elementary physical phenomena. In Einstein's theory of general relativity, mass directly influences curvature but spin has no similar dynamical effect. As a result of the absence of torsion, the infinitesimal holonomy groups of the Cartan connection of an Einstein space consist of only homogeneous transformations [6]. By introducing torsion and relating it to spin, one obtains an interesting link between the theory of gravitation and the theory of special relativity [7].

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*) The research reported in this note was carried out in the spring of 1971, during the author's stay at the Laboratory for Astrophysics and Space Research of the Enrico Fermi Institute for Nuclear Studies, University of Chicago.

It is conceivable that torsion may produce observable effects inside some of the recently discovered astronomical objects, such as the neutron stars, which probably have very strong magnetic fields accompanied by a substantial average value of density of spin.

For a body with given values of spin and mass, the dimensional numbers characterizing the order of magnitude of the effects of torsion and of curvature are, respectively,

$$\text{spin}/(\text{radius})^2 \quad \text{and} \quad \text{mass}/\text{radius}.$$

(We use a system of units in which the gravitational constant and the velocity of light are equal to 1.) For an electron, the ratio of these two (very small) numbers is of the order of $1/a \approx 137$; the influence of spin on geometry is larger than that of mass. This is no longer so for matter in bulk because mass is essentially additive whereas in most circumstances spins cancel out one another.

2. Notation. The model of space-time is assumed to be a four-dimensional differential manifold X of class C^∞. All maps and fields on X are also restricted to be of this class. A local section of the bundle of *linear* frames of X defines 4 fields of 1-forms θ^i $(i=1, \ldots, 4)$ which are linearly independent at each point of X [6]. All geometric objects on X, other than forms, will be described by their components with respect to (θ^i); e.g., the metric tensor on X is written as

$$g_{ij}\, \theta^i \otimes \theta^j, \qquad (i, j = 1, \ldots, 4),$$

and ω^i_j are the 1-forms of a linear connection on X, $\nabla\theta^i = -\omega^i_j \otimes \theta^j$. Under the replacement of θ^i by $\bar{\theta}^i$, where $\theta^i = a^i_j \bar{\theta}^j$ and $(a^i_j) = a\colon X \to GL(4, \boldsymbol{R})$ the connection forms change according to

$$a^i_k\, \bar{\omega}^k_j = \omega^i_k\, a^k_j + \mathrm{d}a^i_j \tag{1}$$

and the components of the metric become

$$\bar{g}_{ij} = g_{kl}\, a^k_i\, a^l_j. \tag{2}$$

Let $\sigma\colon \mathrm{GL}(4, \boldsymbol{R}) \to \mathrm{GL}(N, \boldsymbol{R})$ be a homomorphism of Lie groups and $\sigma'\colon \mathcal{L}(\boldsymbol{R}^4) \to \mathcal{L}(\boldsymbol{R}^N)$ the derived homomorphism of Lie algebras. The linear map σ' may be represented by the matrix (σ^{Bj}_{Ai}), $A, B = 1, \ldots, N$. If $\varphi = (\varphi_A)$ is a tensor-valued p-form of type σ, defined on X*), its covariant exterior derivative [9] with respect to the connection (ω^i_j) is a $(p+1)$-form of type σ, given by

$$\mathrm{D}\varphi_A = d\varphi_A + \sigma^{Bj}_{Ai}\, \omega^i_j \wedge \varphi_B.$$

For a tensor-valued 0-form (φ_A), $D\varphi_A = \theta^i \nabla_i \varphi_A$ is the usual covariant derivative, whereas for a scalar-valued p-form φ, $D\varphi$ reduces to the exterior derivative $d\varphi$.

The curvature and the torsion are 2-forms of type ad, given, respectively, by

$$\Omega^i_j = \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j \quad \text{and} \quad \Theta^i = \mathrm{d}\theta^i + \omega^i_j \wedge \theta^j.$$

*) More precisely, (φ_A) is the pull-back, by that local section of the bundle $L(X)$ of linear frames which is associated to (θ^i), of a horizontal p-form of type σ, defined on $L(X)$ [8].

The forms (θ^i, ω^k_l) and (Θ^i, Ω^k_l) may be considered as pull-backs, from the bundle of *affine* frames to the manifold X, of the Cartan connection and its curvature, respectively [10].

The metric tensor is assumed to have the normal hyperbolic signature so that $g=\det(g_{ij})$ is negative. It is convenient to introduce the completely antisymmetric pseudotensor η_{ijkl}, $\eta_{1234}=\sqrt{-g}$, the tensor-valued forms

$$\eta_{ijk}=\theta^l\,\eta_{ijkl}\,,\qquad \eta_{ij}=\tfrac{1}{2}\,\theta^k\wedge\eta_{ijk}\,,$$
$$\eta_i=\tfrac{1}{3}\,\theta^j\wedge\eta_{ij}\,,\qquad \eta=\tfrac{1}{4}\,\theta^i\wedge\eta_i\,,$$

and to write

$$\eta^i=g^{ij}\,\eta_j\,,\qquad \eta^i_j=g^{ik}\,\eta_{kj}\,,\qquad \eta^{ij}=g^{ik}\,g^{jl}\,\eta_{kl}\,,\qquad \text{etc.}$$

In general, indices are lowered by means of the matrix (g_{ij}) and raised by means of its inverse (g^{ij}), according to the rules of tensor calculus. Clearly, the forms η, η^i, η^{ij}, η^{ijk} and η^{ijkl} are the duals of 1, θ^i, $\theta^i\wedge\theta^j$, $\theta^i\wedge\theta^j\wedge\theta^k$ and $\theta^i\wedge\theta^j\wedge\theta^k\wedge\theta^l$, respectively.

It is easy to prove the following

LEMMA. *If* (λ^i_j) *is a* 1*-form of type* ad *and*

$$\lambda^i_k\wedge\eta^k_j-\eta^i_k\wedge\lambda^k_j=0\,,$$

then there exists a scalar-valued 1*-form* λ *such that* $\lambda^i_j=\delta^i_j\,\lambda$.

3. The field equations in empty space. Let us consider a space-time X with a metric and a linear connection which, to begin with, are unrelated to each other. The 4-form

$$8\pi K=\tfrac{1}{2}\,\eta^k_j\wedge\Omega^j_k$$

is independent of the choice of the frames (θ^i) and, therefore, is defined globally on X. On a Riemannian space-time, the form K reduces to the integrand of the variational principle used to derive Einstein's equations.

By varying the metric, the frames and the linear connection independently of one another, one obtains

$$8\pi\delta K=\tfrac{1}{2}\,E^{ij}\,\delta g_{ij}+e_i\wedge\delta\theta^i-\tfrac{1}{2}\,c^j_i\wedge\delta\omega^i_j+\text{an exact form},$$

where

$$E^{ij}=\tfrac{1}{2}\,(g^{ij}\,\eta^k_l-g^{ik}\,\eta^j_l-g^{jk}\,\eta^i_l)\wedge\Omega^l_k$$

is the (generalized) Einstein tensor (-valued 4-form),

$$e_i=-\tfrac{1}{2}\,\eta^k_{ji}\wedge\Omega^j_k\,,\qquad\text{and}\qquad c^i_j=\mathrm{D}\eta^i_j\,.$$

The Einstein tensor is symmetric, even in the non-Riemannian case, whereas $e_i=g_{ij}\,e^j$ is not, $e^i\wedge\theta^j\neq e^j\wedge\theta^i$.

According to (1) and (2), a variation of the frames $\delta\theta^i=-a^i_j\,\theta^j$ induces the following changes in the connection forms and the components of the metric, the connection and the metric themselves being kept fixed,

$$\delta\omega^i_j=\mathrm{D}\,a^i_j\,,\qquad \delta g_{ij}=a_{ij}+a_{ji}\,.$$

These changes do not affect K; the equation $\delta K=0$ yields the identity

(3) $$E_i^j = e_i \wedge \theta^j + \tfrac{1}{2}\, \mathrm{D} c_i^j .$$

From the principle of least action, $\delta \int K = 0$, by varying with respect to (g_{ij}, ω_l^k) and (θ^i, ω_l^k), one obtains two sets of equations

(4) $$E^{ij}=0, \quad c_k^l=0,$$

and

(5) $$e_i=0, \quad c_k^l=0.$$

It follows from Eq. (3) that these two sets are equivalent to each other.

The lagrangian form K is invariant under the 'projective transformation' of the connection, i.e. under the replacement of ω_j^i by $\omega_j^i+\delta_j^i\,\lambda$, where λ is any 1-form on X. This implies the identity $c_k^k=0$ and makes it impossible to determine, in a unique manner, the connection from Eqs. (4) or (5). By using the Lemma, one proves

THEOREM 1. *For any metric tensor* (g_{ij}), *the equation* $c_k^l=0$ *is equivalent to*

$$\omega_j^i=\gamma_j^i+\delta_j^i\,\lambda$$

where γ_f^i *are the forms of the Riemannian connection of* (g_{ij}) *and* λ *is a* 1-*form.*

This leads to the following

COROLLARY, *The following three conditions are equivalent to one another*: (a) $\omega_j^i=\gamma_j^i$, (b) $c_k^l=0$ *and* $\Theta^i=0$, (c) $c_k^l=0$ *and* $Dg_{ij}=0$.

The equivalence of (a) and (b) is due to Palatini [11].

4. A classical field interacting with gravitation. A classical field, such as the electromagnetic field, may be regarded as a model of a physical system interacting with gravitation. In this case, the equations of motion may be derived from a principle of least action and their formal properties analyzed in detail. To describe the field, let us consider a tensor-valued p-form (φ_A) of type σ and assume a lagrangian 4-form L depending, in a local manner, on g_{ij}, θ^k, φ_A and $D\varphi_A$. Independent variation of the variables leads to

$$\delta L=\tfrac{1}{2}\,T^{ij}\,\delta g_{ij}+t_i\wedge\delta\theta^i-\tfrac{1}{2}\,s_i^j\wedge\delta\omega_j^i+L^A\wedge\delta\varphi_A+\text{an exact form}.$$

If all the variations are induced by a mere change of the frames, then $\delta L=0$ and an argument similar to the one used in the preceding section leads to the identity

(6) $$T_i^j=t_i\wedge\theta^j+\tfrac{1}{2}\,\mathrm{D}s_i^j+\sigma_{Ai}^{Bj}\,L^A\wedge\varphi_B .$$

By varying the total action $\int(K+L)$ with respect to $(\varphi_A, g_{ij}, \omega_l^k)$ and $(\varphi_A, \theta^i, \omega_l^k)$, one obtains two sets of equations

(7) $$L^A=0, \quad E^{ij}=-8\pi\,T^{ij}, \quad c_k^l=-8\pi s_k^l$$

and

(8) $$L^A=0, \quad e_i=-8\pi t_i, \quad c^l_k=-8\pi s^l_k,$$

which are equivalent to each other by virtue of the identities (3) and (6). This shows that, to write the field equations in the Einstein—Cartan theory, one is free to use either the symmetric, tensor-valued 4-form of energy and momentum (T^{ij}) or the 'canonical', asymmetric, vector-valued 3-form (t_i).

The electromagnetic potential should be defined as a scalar-valued 1-form φ so that the field be $F=d\varphi$. The alternative identification of the potential with a covector-valued 0-form (φ_i) would lead to the field $(D\varphi_i)\wedge\theta^i$ which is not gauge-invariant in the presence of torsion. Any scalar-valued form leads to $s^j_i=0$ and $T^j_i=t_i\wedge\theta^j$. Therefore, a pure electromagnetic field cannot be the source of torsion, a fact which is hardly surprising if one remembers the non--local character of the spin of a photon. It is amusing to note that the lagrangian of both the massless scalar and the electromagnetic fields can be represented by one formula, $8\pi L= =-{*}(d\varphi)\wedge d\varphi$, where star denotes the dual of a form and φ is a 0-form (scalar theory) or a 1-form (electromagnetism).

5. The metric theory. The relevance of spinors in physics indicates that the linear connection on space-time is compatible with the metric tensor, i.e., that $Dg_{ij}=0$. Otherwise, there would be no natural lift of the linear connection to a connection on the bundle of spinor frames. By assuming that the linear connection is metric, as was done by Cartan and the other authors [3—5], it is possible to remove the freedom of projective transformations, inherent in the non-metric theory of section 3. The Lemma of section 2 is useful in the proof of

THEOREM 2. *Let* (g_{ij}) *be a metric tensor and let* (s^l_k) *be a 3-form of type ad, defined on X. Among the linear connections on X, satisfying the Cartan equation*

$$c^l_k=-8\pi s^l_k$$

there is exactly one such that

(9) $$Dg_{ij}=0$$

if and only if

(10) $$s_{kl}+s_{lk}=0.$$

The metric conditions (9) and (10), which are assumed to hold from now on, imply

$$D\eta_{ijkl}=0, \quad \Omega_{ij}+\Omega_{ji}=0$$

and lead to the following simple form of the *Einstein—Cartan equations*

(11) $$e_i=\tfrac{1}{2}\eta_{ijk}\wedge\Omega^{jk}=-8\pi t_i,$$

(12) $$c_{ij}=\eta_{ijk}\wedge\Theta^k=-8\pi s_{ij}.$$

By writing $\Theta^k=\frac{1}{2}Q^k_{ij}\,\theta^i\wedge\theta^j$, $s_{ij}=s^k_{ij}\,\eta_k$ and introducing the forms $\Theta_{ij}= =Q^k_{ij}\,\eta_k$, $s^k=\frac{1}{2}s^k_{ij}\,\theta^i\wedge\theta^j$, the last equation can be solved with respect to the components of torsion,

$$\Theta_{ij}=-8\pi\eta_{ijk}\wedge s^k.$$

In the approximation of special relativity, there exists a radius-vector, i.e. a vector field (x^i) such that $Dx^i = \theta^i$ *). Eq. (6), together with the conservation law of energy and momentum, $Dt_i = 0$, and the symmetry of T^{ij} gives rise to the conservation law of total angular momentum:

$$\text{if} \quad L^A = 0, \quad \text{then} \quad D(x^i t^j - x^j t^i + s^{ij}) = 0.$$

The forms $(x^i t^j - x^j t^i)$ and (s^{ij}) are interpreted as the density of orbital angular momentum and of spin, respectively.

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А. Траутман, **Об уравнениях Эйнштейна-Картана. I.**

Содержание. Автор представляет в настоящей работе новые аргументы с целью подтверждения идеи Картана, касающейся обобщения теории Эйнштейна путем установления связи между кручением и внутренним угловым моментом. Показано, что уравнение Эйшнтейна может быть записано в двух эквивалентных формах, с симметрической либо с канонической тензорной плотностью энергии-импульса в качестве источника. Уравнение Картана определяет линейную связность с точностью до проективных преобразований; эта произвольность может быть устранена требованием, чтобы связность была-бы метрической.

*) Incidentally, the existence of a radius-vector field in a space without curvature is equivalent to the vanishing of torsion: $Dx^i = \theta^i$ implies $\Omega^i_j x^j = \Theta^i$.