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MATHEMATICS
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Riemannian Bundles

by

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1. Introduction

In differential geometry and theoretical physics one often considers bundles with a Riemannian space as the typical fibre. Since bundles are usually described by an adjective referring to the structure of the fibres, we propose to call these Riemannian bundles. This notion generalizes that of a Euclidean vector bundle exemplified by the tangent bundle of a Riemannian space. Both vector and Riemannian bundles are special cases of what we call A-bundles, i.e., bundles with fibres belonging to a category A of differential manifolds (cf. Sec. 2). Morphisms of A-bundles are defined in Sec. 3 and a simple example is given to show that an A-bundle may be trivial as a (differential) bundle without being isomorphic to a product in A. The construction of A-bundles associated to a principal bundle is carried over from the theory of vector bundles (Sec. 4). The last section is devoted to the problem of completing the structure of a Riemannian bundle to that of a Riemannian space. The Riemannian space of the Kaluza-Klein theory of electromagnetism is a particular case of such an extension [1].

All the objects and morphisms considered in this paper belong to **Man**, the category of real, finite-dimensional differential manifolds of class C^∞. To alleviate the language, the adjective 'differentiable' is omitted, except in places where this could lead to misunderstanding. We use, with minor modifications, the prevalent terminology and notation of differential geometry [2—4]. Some of the terms and symbols used in this paper are explained in the following paragraph.

A bundle (E, X, π) is characterized by the surjective projection $\pi: E \to X$; a convenient abuse of language consists in referring to E as the bundle over X. The tangent functor is denoted by T. The kernel of $T\pi: TE \to TX$ is the *vertical bundle* ver E over E, consisting of vertical vectors, i.e., vectors tangent to the fibres of E. The quotient of TE by ver E is the *horizontal bundle* hor E over E. The fibre of hor E over $p \in E$ is the vector space $\mathrm{hor}_p E = T_p E/\ker T_p \pi$. The inclusion i of ver E in TE, together with the canonical surjection s of TE on hor E, yields the exact sequence of morphisms of vector bundles over E

$$(1) \qquad 0 \longrightarrow \mathrm{ver}\, E \xrightarrow{\ i\ } TE \xrightarrow{\ s\ } \mathrm{hor}\, E \longrightarrow 0$$

Moreover, $T\pi$ factors through hor E, $T\pi = t \circ s$, and, for any $p \in E$, $t \mid \mathrm{hor}_p E$ is a linear isomorphism onto $T_{\pi(p)} X$. A Riemannian space (X, k) is defined by a function k on the tangent bundle TX of the manifold X, k being such that $k_x = k \mid T_x X$ is a non-degenerate quadratic form on the tangent space $T_x X$ at $x \in X$. The signature of k_x is assumed to be the same all over X but otherwise arbitrary, so that the adjective "Riemannian" means here what is often referred to as "pseudo-Riemannian". The vector space associated to an affine space E is denoted by $\vec{E}$; the additive group of $\vec{E}$ acts freely and transitively on E. For an affine space E, the tangent bundle may and will be identified with the product $E \times \vec{E}$.

2. Definition and examples

DEFINITION. A bundle (E, X, π) is called an *A-bundle*, where $\boldsymbol{A}$ is a subcategory of **Man** if

(a) the typical fibre Y of E and all fibres $E_x = \pi^{-1}(x)$, $x \in X$, are objects of $\boldsymbol{A}$, and

(b) for any $x' \in X$ there exists an open neighbourhood U of x' and a differential isomorphism ("diffeomorphism") $\varphi: U \times Y \to \pi^{-1}(U)$ such that, for every $x \in U$, the map $y \mapsto \varphi_x(y) = \varphi(x, y)$ is an $\boldsymbol{A}$-isomorphism of Y on E_x.

By taking for $\boldsymbol{A}$ the category of Riemannian spaces, the category of Euclidean spaces, the category $\boldsymbol{V}$ of vector spaces or the category $\boldsymbol{EV}$ of Euclidean vector spaces, one obtains, respectively, the notion of *Riemannian bundles, Euclidean bundles, vector bundles* or *Euclidean vector bundles* *).

In particular, with a Riemannian bundle (E, X, π) there is associated a Riemannian space (Y, l) and the *fibre metric* $g: \mathrm{ver}\, E \to \boldsymbol{R}$ such that, for any $x \in X$, there is a local trivialization $\varphi: U \times Y \to \pi^{-1}(U)$, $x \in U$, satisfying $g \circ T\varphi_x = l$. Clearly, if E is a Riemannian bundle, then ver E is a Euclidean vector bundle over E, but the converse is not true.

Example 1. A *Galilean space* is an affine space E together with a positive-definite scalar product defined on a vector subspace $\vec{Y} \subset \vec{E}$, of codimension 1 in $\vec{E}$. The quotient space $X = E/\vec{Y}$ is affine, $\vec{X} = \vec{E}/\vec{Y}$, and the canonical map $\pi: E \to X$ defines a Euclidean bundle with typical fibre $\vec{Y}$. In Newtonian mechanics, E is four-dimensional and X is called the *absolute time* [5].

Example 2. If E is a bundle over a Riemannian space (X, k), then hor E has the structure of a Euclidean vector bundle over E, defined by $k \circ t$.

Example 3. The tangent bundle TX of a Riemannian space X is a Euclidean vector bundle. The subbundle E of TX, consisting of all tangent vectors of unit length, is a Riemannian bundle over X with a space of constant curvature as the typical fibre. In particular, if X is a time-oriented $(n+1)$-dimensional Lorentz space [6], then there exists a subbundle $E_1 \subset E$, consisting of future-pointing, time-like vectors of unit length. The bundle E_1 over X is Riemannian and its typical fibre is

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*) Some authors use the term 'Riemannian vector bundle' for what we call a Euclidean vector bundle, [cf. 2, vol. II, p. 315].

the n-dimensional Lobatchevski space. This bundle plays a role in relativistic kinematics.

Example 4. Let (E, G, X, π, ψ) be a principal bundle with structure group G and projection $\pi: E \to X$. The Lie group G acts in E on the right, $\psi(p, a) = \psi_a(p) = pa$, $\psi_e = id$, $\psi_a \circ \psi_b = \psi_{ba}$, where $p \in E$ and $ae = a$, $b \in G$. The left and right translations in G are denoted, respectively, by γ and δ: $\gamma_a(b) = ab = \delta_b(a)$. The map $\psi_p: G \to \to E_{\pi(p)}$ defined by $\psi_p(a) = pa$ is a differential isomorphism and may be used to transport any tensor field from G to $E_{\pi(p)}$. From

$$\psi_{pa} = \psi_p \circ \gamma_a$$

it follows that the image by ψ_p of a left-invariant tensor field on G is independent of the choice of p within the fibre $E_{\pi(p)}$. On the other hand, if the field on G is right-invariant, then by virtue of

$$\psi_a \circ \psi_p = \psi_p \circ \delta_a$$

the transported field is invariant with respect to the action of G on the fibre. Therefore, a biinvariant metric tensor field on G, such as the one given by the Killing form on a semi-simple Lie group, defines on E the structure of a Riemannian bundle, invariant with respect to the action of G.

3. Morphisms

DEFINITION. Let (E, X, π) and (E', X', π') be A-bundles; the differentiable map $F: E \to E'$ is called an A-bundle morphism if there exists a differentiable map $f: X \to X'$ such that F restricted to any fibre E_x is an A-morphism of E_x in $E'_{f(x)}$.

This definition makes the class of all A-bundles into a category, denoted by $\boldsymbol{AB}$. If X is a manifold and Y an object of $\boldsymbol{A}$, then $(X \times Y, X, pr_1)$ is an A-bundle, called the *product A-bundle*. Any bundle $\boldsymbol{AB}$-isomorphic to such a product is called a *trivial A-bundle*. The following example shows that a Euclidean vector bundle may be a product bundle without being trivial in the category $\boldsymbol{EVB}$. This gives rise to new problems of classification.

Example. Let $x \mapsto \bar{x} = \{x' \in \boldsymbol{R}: x' - x \in \boldsymbol{Z}\}$ denote the canonical map of $\boldsymbol{R}$ on the one-dimensional torus $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ and put $E = \boldsymbol{T} \times \boldsymbol{R}^2$. The manifold E may be given the structure of a Euclidean vector bundle over $\boldsymbol{T}$ by introducing in the vector fibre $E_{\bar{x}} = \{(\bar{x}, y, z): y, z \in \boldsymbol{R}\}$ a scalar product defined by the quadratic form in y and z:

$$2yz \cos 2\pi x + (y^2 - z^2) \sin 2\pi x.$$

E is not a trivial $\boldsymbol{EV}$-bundle because the "null" subspace $\{(\bar{x}, t \sin \pi x, t \cos \pi x): x, t \in \boldsymbol{R}\}$ is homeomorphic to the Möbius band.

A functor $\Phi: \boldsymbol{A}_1 \to \boldsymbol{A}_2$ from one subcategory of **Man** to another can be extended to the functor $\Phi\boldsymbol{B}: \boldsymbol{A}_1\boldsymbol{B} \to \boldsymbol{A}_2\boldsymbol{B}$ of the same variance as Φ. A similar proposition is true for functors of several variables. If $\Psi: \boldsymbol{A}_2 \to \boldsymbol{A}_3$ is another functor, then so is $\Psi \circ \Phi$ and $(\Psi \circ \Phi)\boldsymbol{B} = \Psi\boldsymbol{B} \circ \Phi\boldsymbol{B}$. For example, any vector functor, such as duality or the tensor product, extends to a functor on the category of vector bundles; the functor TB is equivalent, by a natural transformation, to the functor ver which assigns the bundle ver E over X to the bundle E over X. If Φ is the Cartesian product of manifolds, then ΦB is the Cartesian product of bundles.

4. Associated bundles and connections

A general method of constructing A-bundles is described by the following

PROPOSITION 1. *If (E, G, X, π, ψ) is a principal bundle, Y an object of A and $\sigma: G \to \mathrm{Aut}_A Y$ a homomorphism of the group G into the group of A-automorphisms of Y, then the bundle $(E_\sigma, X, \pi_\sigma)$, associated to E by the representation σ, may be given the structure of an A-bundle.*

The proof is based on the standard construction of the associated bundle [2]. Let the action of G be extended to $E \times Y$ by $(p, y)\, a^{-1} = (pa^{-1}, \sigma_a(y))$ and let κ be the canonical map of $E \times Y$ on the quotient space $E_\sigma = (E \times Y)/G$. The manifold E_σ may be made into a bundle over X; the projection $\pi_\sigma: E_\sigma \to X$ is well defined by $\pi_\sigma \circ \kappa = \pi \circ pr_1$ because $\kappa_{pa} = \kappa_p \circ \sigma_a$, where $\kappa_p(y) = \kappa(p, y)$. If $q: U \to E$ is a local section, $\pi \circ q = id$, then the map $(x, y) \mapsto \kappa(q(x), y)$ is a differential isomorphism of $U \times Y$ onto $\pi_\sigma^{-1}(U)$. The bijection $\kappa_{q(x)}$ of Y onto $E_{\sigma x} = \pi_\sigma^{-1}(x)$ can be used to make $E_{\sigma x}$ into an object of A by requiring that $\kappa_{q(x)}$ be an A-isomorphism. By taking a different local section q', defined in a neighbourhood of x, one obtains the same A-structure on $E_{\sigma x}$, because, if $q'(x) = q(x)\, a$, then $\kappa_{q'(x)} = \kappa_{q(x)} \circ \sigma_a$, and σ_a is an A-automorphism,

A *horizontal differential system* on a bundle (E, X, π) is a splitting of the sequence (1) defined by a VB-morphism

$$h: \mathrm{hor}\, E \to TE, \qquad s \circ h = \mathrm{id},$$

or, equivalently, by the VB-morphism

$$v: TE \to \mathrm{ver}\, E, \qquad v \circ i = \mathrm{id},$$

with

$$v \circ h = 0 \quad \text{and} \quad i \circ v + h \circ s = \mathrm{id}.$$

A horizontal differential system on E defines the structure of an almost product manifold [7]. The vector space $h(\mathrm{hor}_p E)$ is called the *horizontal subspace* at $p \in E$; v is the *vertical projection* corresponding to h.

The action of G in the principal bundle E extends to TE. Because of $\pi \circ \psi_a = \pi$, the action of G in TE induces an action of the group in both ver E and hor E. The corresponding maps are denoted, respectively, by ver ψ_a and hor ψ_a,

$$i \circ \mathrm{ver}\, \psi_a = T\psi_a \circ i, \qquad \mathrm{hor}\, \psi_a \circ s = s \circ T\psi_a.$$

A *connection* in the principal bundle E is a horizontal differential system h on the bundle E, equivariant with respect to G,

$$h \circ \mathrm{hor}\, \psi_a = T\psi_a \circ h.$$

The corresponding vertical projection is also equivariant,

(2) $$v \circ T\psi_a = \text{ver}\, \psi_a \circ v\,.$$

The canonical map κ of $E\times Y$ onto the bundle E_σ associated to E induces vector bundle morphisms: ver κ of ver $E\times TY$ onto ver E_σ and hor κ of hor $E\times TY$ onto hor E_σ. More precisely, the triple (ver κ, $T\kappa$, hor κ) defines a morphism of the exact sequence

$$0 \to \text{ver}\, E\times TY \to TE\times TY \to \text{hor}\, E\times TY \to 0$$

on the sequence

$$0 \longrightarrow \text{ver}\, E_\sigma \xrightarrow{i_\sigma} TE_\sigma \xrightarrow{s_\sigma} \text{hor}\, E_\sigma \longrightarrow 0\,.$$

If the principal bundle E has a connection h, then a bundle E_σ associated to E can be endowed with the horizontal differential system h_σ: hor $E_\sigma \to TE_\sigma$, $s_\sigma \circ h_\sigma = id$, defined in a natural way by h,

$$h_\sigma \circ \text{hor}\, \kappa = T\kappa \circ (h\times \text{id})\,.$$

5. Riemannian metrics on Riemannian bundles

Let (E, X, π) be a Riemannian bundle with fibre metric g; a Riemannian metric $\tilde{g}$ on E is called an *extension* of g if $\tilde{g}\circ i = g$. We shall also say that the Riemannian metric $\tilde{g}$ is *compatible* with the structure of the Riemannian bundle E. An obvious question to ask is whether any Riemannian bundle admits a Riemannian extension. To show that this is so, consider a horizontal differential system h on E, the corresponding vertical projection v and a Riemannian metric k on X. The formula

(3) $$\tilde{g} = k\circ T\pi + g\circ v$$

defines an extension of g. Since for any paracompact manifold X and any bundle E over X there exist both a Riemannian metric on X and a splitting of (1), we obtain the following

PROPOSITION 2. *Any Riemannian bundle over a paracompact manifold admits a Riemannian metric compatible with the structure of the bundle.*

Example. A horizontal differential system h on the Galilean bundle E is called an *ether*. The scalar product on $\vec{Y}$ defines the fibre metric g: ver $E\to \mathbf{R}$. The absolute time X may be measured by means of $k = -\tau\otimes\tau$, where $\tau \in \vec{E}^*$ is a form whose kernel coincides with $\vec{Y}$. The Lorentz metric on E, given by Eq. (3), constitutes an essential element of pre-relativistic electrodynamics. If h is invariant with respect to $\vec{E}$, then the ether is said to be rigid and $\tilde{g}$ is the Minkowski (flat) metric.

According to Proposition 1, if (Y, l) is a Riemannian space and $\sigma: G\to \text{Aut}\, Y$ is a homomorphism of G into the group of isometries of Y, $l\circ T\sigma_a = l$, then the bundle E_σ associated to (E, G, X, π, ψ) admits a fibre metric g determined by $g\circ \text{ver}\, \kappa = l\circ pr_2$. Moreover, if (X, k) is a Riemannian space, E has a connection h and v_σ is the corresponding vertical projection on E_σ, then

(4) $$\tilde{g} = k\circ T\pi_\sigma + g\circ v_\sigma$$

is a Riemannian metric on E_σ. For example, if (a) X is an n-dimensional proper Riemannian space, (b) E is the bundle of orthonormal frames of X, with the Riemannian connection h, (c) l is the canonical metric on $Y=\boldsymbol{R}^n$, and (d) σ is the defining representation of $G=O(n)$ in $\boldsymbol{R}^n$, then E_σ may be identified with TX and Eq. (4) gives the natural Riemannian metric on the tangent bundle of the Riemannian space [8].

The bundle associated to (E, G, X, π, ψ) by the representation of G by left translations can be identified with E itself. In this case, the map κ coincides with $\psi: E\times G\to E$ and $\mathrm{ver}\,\psi: \mathrm{ver}\,E\times TG\to \mathrm{ver}\,E$ composed with the obvious injection $E\times T_e G\to \mathrm{ver}\,E\times TG$ determines the isomorphism $\lambda: \mathrm{ver}\,E\to E\times T_e G$ of vector bundles over E, equivariant with respect to G,

$$\lambda\circ \mathrm{ver}\,\psi_a=(\psi_a\times T\,\mathrm{ad}_a)\circ\lambda \tag{5}$$

where $\mathrm{ad}_a(b)=a^{-1}ba$. If v is the vertical projection corresponding to a connection in E, then $\omega=pr_2\circ\lambda\circ v$ is the *connection form* and Eqs. (2) and (5) imply

$$\omega\circ T\psi_a=T\,\mathrm{ad}_a\circ\omega\,.$$

If l is a left-invariant metric on G, then the fibre metric g defined by $g\circ \mathrm{ver}\,\psi= =l\circ pr_2$ is equal to $l\circ pr_2\circ\lambda$. If, in addition, l is right-invariant, then $l\circ T\,\mathrm{ad}_a=l$ and the extension $\tilde g=k\circ T\pi+l\circ\omega$, called by Gray [7] the *natural metric* on E, is invariant with respect to G, $\tilde g\circ T\psi_a=\tilde g$ for any $a\in G$. If (X, k) is interpreted as space-time and G as the 'gauge' group in a theory of the type considered by Yang, Mills and Utiyama [9], then $(E, \tilde g)$ becomes the analogue of the five-dimensional Riemannian space introduced by Kaluza and Klein to describe geometrically the electromagnetic field.

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