REFERENCES

[1] A. Sommerfeld, Math. Ann. **47** (1896), 317.
[2] — Zeitschr. f. Math. u. Phys. **46** (1901), 11.
[3] W. Rubinowicz, Ann. Phys. **53** (1917), 257.
[4] T. J. I. A. Bromwich, Proc. Lond. Mat. Soc. **14** (1915), 450.
[5] P. C. Clemmov, Proc. Roy. Soc., A, **205** (1950), 286.
[6] T. B. A. Senior, Quart J. Mech. Appl. Math. IV **1** (1953), 101.
[7] I. V. Vandakurov, Jurn. Eksp. i Teoret. Fiz. (in Russian) **26** (1954), 3.
[8] R. Teisseyre, *The diffraction of a dipole field by a perfectly conducting wedge*, Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 157.
[9] — *The general solution for the diffraction of a dipole field by a perfectly conducting wedge*, Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 523.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

THEORETICAL PHYSICS

On a Generalisation of the Einstein-Infeld Approximation Method

by

A. TRAUTMAN

Presented by L. INFELD on April 9, 1956

1. The new approximation method formulated first by Einstein, Infeld and Hoffman [1], and developed in further works of Einstein, Infeld [2]-[5], Robertson [6], Papapetrou [7] and others (for references see [8]) is successfully used to derive the equations of motion in the general relativity theory. The method is based on expanding all functions (the metric tensor $g_{\alpha\beta}$, the energy-momentum tensor $T_{\alpha\beta}$) into Taylor series in the parameter $\lambda=1/c$ (c=velocity of light) and on introducing an auxiliary time $\tau=\lambda x^0$. In the papers quoted above, it was assumed that these series contain terms corresponding either to odd or to even powers of λ, e. g., $g_{00}=\sum_{l=0}^{\infty}\lambda^{2l}\underset{2l}{g_{00}}$ (as stated by Infeld [5]). A further assumption was that $\underset{0}{g_{\alpha\beta}}$ has the Galilean values

$$\underset{0}{g_{00}}=1,\qquad \underset{0}{g_{0k}}=0,\qquad \underset{0}{g_{mn}}=-\delta_{mn} \tag{1}$$

(Greek indices run from 0 to 3, Latin – from 1 to 3). In this paper it is shown that the first of these assumptions can be removed and the second easily generalised. An application of the modified theory will be given in a subsequent paper.

2. We use the conventional notation of the theory of relativity (e. g., summation convention). The derivatives with respect to x^ν we denote by a stroke $|$, that with respect to τ or x^k by a comma, viz.:

$$\frac{\partial\varphi}{\partial x_\alpha}=\varphi_{|\alpha},\qquad \frac{\partial\varphi}{\partial x^0}=\varphi_{|0}=\lambda\varphi_{,0}=\lambda\frac{\partial\varphi}{\partial\tau},\qquad \frac{\partial\varphi}{\partial x^k}=\varphi_{|k}=\varphi_{,k}.$$

(α) when written under a letter denotes $\delta_{0\alpha}$.

The gravitational field equations we shall write in the form

$$R_{\alpha\beta}=\varkappa(T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T), \tag{2}$$

where

(3) $$R_{\alpha\beta}=\tfrac{1}{2}\,g^{\mu\nu}(g_{\alpha\mu|\beta\nu}+g_{\beta\mu|\alpha\nu}-g_{\alpha\beta|\mu\nu}-g_{\mu\nu|\alpha\beta})+g^{\mu\nu}g^{\sigma\varrho}(\Gamma_{\nu\alpha\sigma}\Gamma_{\mu\varrho\beta}-\Gamma_{\nu\alpha\beta}\Gamma_{\mu\varrho\sigma}),$$

$T_{\alpha\beta}$ denotes the energy-momentum tensor, $T=T^{\alpha}_{\alpha}$, and

$$\Gamma_{\alpha\beta\gamma}=\tfrac{1}{2}(g_{\alpha\beta|\gamma}+g_{\gamma\alpha|\beta}-g_{\beta\gamma|\alpha}).$$

We assume that in the region of the x^{ν}-space, where we look for the field, we can develop $g_{\alpha\beta}$, $g^{\alpha\beta}$ and $T_{\alpha\beta}$ into power series:

(4) $$g_{\alpha\beta}=\sum_{l=0}^{\infty}\lambda^{l}\underset{l}{g}_{\alpha\beta},\qquad g^{\alpha\beta}=\sum_{l=0}^{\infty}\lambda^{l}\underset{l}{g}^{\alpha\beta},\qquad T_{\alpha\beta}=\sum_{l=0}^{\infty}\lambda^{l}\underset{l}{T}_{\alpha\beta}.$$

From $g_{\alpha\nu}g^{\nu\beta}=\delta^{\beta}_{\alpha}$ we obtain

(5) $$\underset{0}{g}_{\alpha\nu}\underset{0}{g}^{\nu\beta}=\delta^{\beta}_{\alpha},$$

(6) $$\sum_{k=0}^{n}\underset{k}{g}_{\alpha\nu}\underset{n-k}{g}^{\nu\beta}=0,\qquad n=1,2,\dots$$

From (5) and (6) we can evaluate $\underset{l}{g}^{\alpha\beta}$ as a function of the $\underset{0}{g}_{\alpha\beta},\underset{1}{g}_{\alpha\beta},\dots,\underset{l}{g}_{\alpha\beta}$ and $\underset{0}{g}^{\alpha\beta}$ (or vice-versa). The result is

(7) $$\underset{l}{g}^{\alpha\beta}=\sum(-1)^{p}\underset{0}{g}^{\alpha\nu_1}\underset{k_1}{g}_{\nu_1\nu_2}\underset{0}{g}^{\nu_2\nu_3}\underset{k_2}{g}_{\nu_3\nu_4}\cdots\underset{k_p}{g}_{\nu_{2p-1}\nu_{2p}}\underset{0}{g}^{\nu_{2p}\beta}.$$

The sum in (7) is to be taken over all (ordered) sets of positive integers $k_1,k_2,\dots,k_p$ such that $k_1+k_2+\dots+k_p=l$. Assuming for $R_{\alpha\beta}$ an expansion similar to that of $T_{\alpha\beta}$, we have

(8) $$\underset{l}{R}_{\alpha\beta}=\tfrac{1}{2}\sum_{k=0}^{l}\underset{l-k}{g}^{\mu\nu}\Big(\underset{k-(\beta)-(\nu)}{g_{\alpha\mu,\beta\nu}}+\underset{k-(\alpha)-(\nu)}{g_{\beta\mu,\alpha\nu}}-\underset{k-(\mu)-(\nu)}{g_{\alpha\beta,\mu\nu}}-\underset{k-(\alpha)-(\beta)}{g_{\mu\nu,\alpha\beta}}\Big)+$$
$$+\tfrac{1}{4}\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{i=0}^{j}\underset{l-k}{g}^{\mu\nu}\underset{k-j}{g}^{\sigma\varrho}\Big[\Big(\underset{j-i-(\sigma)}{g_{\nu\alpha,\sigma}}+\underset{j-i-(\alpha)}{g_{\sigma\nu,\alpha}}-\underset{j-i-(\nu)}{g_{\alpha\sigma,\nu}}\Big)\Big(\underset{i-(\varrho)}{g_{\mu\beta,\varrho}}+\underset{i-(\beta)}{g_{\varrho\mu,\beta}}-\underset{i-(\mu)}{g_{\varrho\beta,\mu}}\Big)+$$
$$-\Big(\underset{j-i-(\beta)}{g_{\nu\alpha,\beta}}+\underset{j-i-(\alpha)}{g_{\nu\beta,\alpha}}-\underset{j-i-(\nu)}{g_{\alpha\beta,\nu}}\Big)\Big(\underset{i-(\sigma)}{g_{\mu\varrho,\sigma}}+\underset{i-(\varrho)}{g_{\mu\sigma,\varrho}}-\underset{i-(\mu)}{g_{\varrho\sigma,\mu}}\Big)\Big].$$

Equation (8) gives, together with (7), an explicit expression of $\underset{l}{R}_{\alpha\beta}$ as a function of the $\underset{k}{g}^{\alpha\beta}$'s and their derivatives. The Einstein equations (2) are equivalent to a set of equations:

(9.$l.\alpha\beta$) $$\underset{l}{R}_{\alpha\beta}=\varkappa\Big(\underset{l}{T}_{\alpha\beta}-\tfrac{1}{2}\sum_{k=0}^{l}\sum_{j=0}^{k}\underset{l-k}{g}_{\alpha\beta}\underset{k-j}{g}^{\mu\nu}\underset{j}{T}_{\mu\nu}\Big),\qquad l=0,1,2,\dots$$

3. Let us assume that in our co-ordinate system the tensor $\underset{0}{g}_{\alpha\beta}$ has the form:

(10) $$\underset{0}{g}_{\alpha\beta}=\left(\begin{array}{c|c}1&0\\\hline 0&\underset{0}{g}_{ik}\end{array}\right),$$

where $\underset{0}{g}_{ik}=\underset{0}{g}_{ik}(x^{r})$, and that the x^{k}-space with the metric $\underset{0}{g}_{ik}$ is (for each $x^{0}=$const.) Euclidean. This means that we assume a flat space-time in the zero-order approximation, and use inertial but in general non-cartesian co-ordinate systems.

From these assumptions we have $\underset{0}{R}_{\alpha\beta}=0$ ($\underset{0}{R}_{ik}=0$ follows from the fact that $\underset{0}{g}_{ik}$ is Euclidean), which implies, by virtue of (9.0. $\alpha\beta$), that $\underset{0}{T}_{\alpha\beta}=0$. The set of functions $\underset{l}{g}_{mn}$ (or $\underset{l}{R}_{mn}$) ($m,n=1,2,3$) forms a covariant tensor with respect to transformations of the 3-dimensional space: $x^{k}\to x^{k'}$ (not involving λ). Likewise $\underset{l}{g}_{0n}$ (or $\underset{l}{R}_{0n}$) is a 3-vector and $\underset{l}{g}_{00}$ (or $\underset{l}{R}_{00}$) is a scalar. We shall denote by a semicolon the covariant differentiation in the x^{k}-space with metric $\underset{0}{g}_{ik}$. From (8) we get

(11) $$\underset{l}{R}_{00}=-\tfrac{1}{2}\underset{0}{g}^{mn}\big(\underset{l}{g}_{00,mn}-\underset{0}{g}^{rs}\underset{0}{g}_{mr,s}\underset{l}{g}_{00,n}+\tfrac{1}{2}\underset{0}{g}^{rs}\underset{0}{g}_{rs,m}\underset{l}{g}_{00,n}\big)+\underset{l}{R'}_{00}$$
$$=-\tfrac{1}{2}\underset{0}{g}^{mn}\underset{l}{g}_{00;mn}+\underset{l}{R'}_{00}=\tfrac{1}{2}\Delta\underset{l}{g}_{00}+\underset{l}{R'}_{00},$$

(12) $$\underset{l}{R}_{0k}=\tfrac{1}{2}\underset{0}{g}^{mn}\big(\underset{l}{g}_{0m;kn}-\underset{l}{g}_{0k;mn}\big)+\underset{l}{R'}_{0k},$$

(13) $$\underset{l}{R}_{ik}=\tfrac{1}{2}\underset{0}{g}^{mn}\big(\underset{l}{g}_{im;kn}-\underset{l}{g}_{ik;mn}+\underset{l}{g}_{nk;mi}-\underset{l}{g}_{mn;ki}\big)-\tfrac{1}{2}\underset{l}{g}_{00;ik}+\underset{l}{R'}_{ik}.$$

The functions $\underset{l}{R'}_{\alpha\beta}$ depend only on $\underset{0}{g}_{\mu\nu},\underset{1}{g}_{\mu\nu},\dots,\underset{l-1}{g}_{\mu\nu}$ and their derivatives, and can be explicitily evaluated from (8). The right-hand side of (9.$l.\alpha\beta$) does not involve $\underset{l}{g}_{\alpha\beta}$, because $\underset{0}{T}_{\alpha\beta}=0$. Thus we see that $\underset{l}{g}_{\alpha\beta}$ enters in (9. $l.$ $\alpha\beta$) linearly and only through the terms written explicitly in (11)-(13). An essential fact is that the equations (9. $l.$ $\alpha\beta$) do not involve the time derivatives of $\underset{l}{g}_{\alpha\beta}$.

Equation (9. $l.$ 00) has a unique solution providing the x^{ν}-space is pseudo-Euclidean at infinity. This is not, however, the case with equations (9. $l.$ αk). If $\underset{l}{g}_{\alpha\beta}$ is a solution of (9. $l.$ $\alpha\beta$), then $\underset{l}{g}^{*}_{\alpha\beta}$ defined by the relations

(14) $$\underset{l}{g}^{*}_{00}=\underset{l}{g}_{00},\qquad \underset{l}{g}^{*}_{0k}=\underset{l}{g}_{0k}+\underset{l}{\varphi}_{0;k},\qquad \underset{l}{g}^{*}_{ik}=\underset{l}{g}_{ik}+\underset{l}{\varphi}_{i;k}+\underset{l}{\varphi}_{k;i},$$

($\underset{l}{\varphi}_{\alpha}$ — arbitrary functions) is also a solution of these equations. This can be easily seen from (11)-(13). The existence of several solutions of the field equations is in conformity with the general principle of relativity;

the transformation $\underset{l}{g}_{\alpha\beta} \to \underset{l}{g}^{*}_{\alpha\beta}$ can be regarded as resulting from a co-ordinate transformation [9].

Assuming certain conditions for the co-ordinate system, e. g.,

$$(15) \qquad \underset{0}{g}^{pq} \underset{l}{g}_{0p;q} = 0, \quad \underset{0}{g}^{pq} (\underset{l}{g}_{kp;q} - \tfrac{1}{2} \underset{l}{g}_{pq;k}) = 0 \quad (k = 1, 2, 3),$$

we can simplify the expression for $R_{i\beta}$, which becomes, in the case of (15):

$$(16) \qquad \begin{aligned} \underset{l}{R}_{\alpha 0} &= -\tfrac{1}{2} \underset{0}{g}^{pq} \underset{l}{g}_{\alpha 0;pq} + \underset{l}{R'}_{\alpha 0}, \\ \underset{l}{R}_{ik} &= -\tfrac{1}{2} \underset{0}{g}^{pq} \underset{l}{g}_{ik;pq} - \tfrac{1}{2} \underset{0}{g}^{q} \underset{l}{g}_{00;ik} + \underset{l}{R'}_{ik}. \end{aligned}$$

We conclude with a simple remark which can be useful in applying the method.

Let us take a flat x^{ν}-space (hence $T_{\alpha\beta} = 0$); this means that there exists a Galilean co-ordinate system in which $g_{\alpha\beta}$ has the form (1). Let us now introduce a non-inertial co-ordinate system $x^{\nu'}$, defined by $\tau = \tau'$ (i. e. $x^0 = x^{0'}$), $x^k = x^k(\tau', x^{k'})$. We have for $g_{\alpha'\beta'}$:

$$g_{0'0'} = g_{\alpha\beta}\, x^{\alpha}_{|0'}\, x^{\beta}_{|0'} = g_{00} + \lambda^2 g_{ik}\, x^{i}_{,0'}\, x^{k}_{,0'} = 1 - \lambda^2 \delta_{ik}\, x^{i}_{,0'}\, x^{k}_{,0'},$$

$$g_{0'k'} = g_{\alpha\beta}\, x^{\alpha}_{|0'}\, x^{\beta}_{|k'} = -\lambda \delta_{ik}\, x^{i}_{,0'}\, x^{k}_{,k'},$$

$$g_{i'k'} = g_{\alpha\beta}\, x^{\alpha}_{|i}\, x^{\beta}_{|k'} = -\delta_{ik}\, x^{i}_{,i'}\, x^{k}_{,k'}.$$

Thus, in a general non-inertial co-ordinate system, the metric tensor $g_{\alpha\beta}$ can be written $g_{\alpha\beta} = \underset{0}{g}_{\alpha\beta} + \lambda \underset{1}{g}_{\alpha\beta} + \lambda^2 \underset{2}{g}_{\alpha\beta}$, where $\underset{0}{g}_{\alpha\beta}$ has the form (10),

$$\underset{1}{g}_{\alpha\beta} = \left(\begin{array}{c|c} 0 & \underset{1}{g}_{0k} \\ \hline \underset{1}{g}_{i0} & 0 \end{array} \right), \quad \text{and} \quad \underset{2}{g}_{\alpha\beta} = \left(\begin{array}{c|c} \underset{2}{g}_{00} & 0 \\ \hline 0 & 0 \end{array} \right).$$

I wish to express my thanks to Professor J. Plebański for suggesting the problem, and for many valuable discussions. I am also indebted to Professor L. Infeld for his kind interest in this work.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] A. Einstein, L. Infeld, B. Hoffmann, Ann. Math. **39** (1938), 66.
[2] A. Einstein, L. Infeld, Ann. Math. **41** (1940), 455.
[3] — Can. J. Math. **1** (1949), 205.
[4] L. Infeld, Acta Phys. Polon. **13** (1954), 187.
[5] — Phys. Rev. **53** (1938), 836.
[6] H. P. Robertson, Ann. Math. **39** (1938), 101.
[7] A. Papapetrou, Proc. Phys. Soc. (London) A **64** (1951), 57.
[8] A. E. Scheidegger, Rev. Mod. Ph. **25** (1953), 451.
[9] L. Infeld, A. E. Scheidegger, Can. J. Math. **3** (1951), 195.

BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV. No. 7. 1956

THEORETICAL PHYSICS

Solution of One-Body Problem by the Einstein-Infeld Approximation Method

by

A. TRAUTMAN

Presented by L. INFELD on April 9, 1956

In the previous paper [1] we presented a generalisation of the "new approximation method". In this paper we shall use that method to evaluate the gravitational field of a point mass, using the notation of [1]. We define $T_{\alpha\beta}$ so that $\varkappa = 8\pi k$, where $k = 6.67 \cdot 10^{-8}$cm.^{3} g^{-1} sec.$^{-2}$ (it is perhaps more usual to put $\varkappa = 8\pi\, kc^{-4}$). The energy-momentum tensor for a point mass will be represented by an expression involving the three-dimensional Dirac δ-function. This method of representing singularities was introduced by Infeld [2].

We assume

$$(1) \qquad \underset{0}{g}_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

and denote

$$x^1 = r, \quad x^2 = \vartheta, \quad x^3 = \varphi.$$

Further,

$$(2) \qquad T^0_0 = \frac{1}{c^4} mc^2 \delta(\vec{r}) = m\lambda^2 \delta(\vec{r}) = \lambda^2 \underset{2}{T}{}^0_0,$$

$$T^{\alpha}_{\beta} = 0 \quad \text{if} \quad \alpha + \beta \neq 0.$$

We also assume that:

(3) *the metric $g_{\alpha\beta}$ is pseudo-Euclidean at infinity;*

(4) $\underset{l}{g}_{22} = \underset{l}{g}_{33} = \underset{l}{g}_{12} = \underset{l}{g}_{13} = \underset{l}{g}_{23} = 0$ and $\underset{l}{g}_{11,2} = \underset{l}{g}_{11,3} = 0$ ("spherical symmetry").