On a Generalisation of the Einstein-Infeld Approximation Method

by

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1. The new approximation method formulated first by Einstein, Infeld and Hoffman [1], and developed in further works of Einstein, Infeld [2]-[5], Robertson [6], Papapetrou [7] and others (for references see [8]) is successfully used to derive the equations of motion in the general relativity theory. The method is based on expanding all functions (the metric tensor $g_{ab}$, the energy-momentum tensor $T_{ab}$) into Taylor series in the parameter $\lambda = 1/c$ ($c$ = velocity of light) and on introducing an auxiliary time $\tau = 1/\lambda$. In the papers quoted above, it was assumed that these series contain terms corresponding either to odd or to even powers of $\lambda$, e. g.

$$g_{ab} = \sum_{\ell=0}^{\infty} \lambda^{2\ell} g_{ab}^{(2\ell)}$$

(as stated by Infeld [5]). A further assumption was that $g_{ab}$ has the Galilean values

$$g_{00} = 1, \quad g_{0\alpha} = 0, \quad g_{\alpha\alpha} = -\delta_{\alpha\alpha}$$

(Greek indices run from 0 to 3, Latin – from 1 to 3). In this paper it is shown that the first of these assumptions can be removed and the second easily generalised. An application of the modified theory will be given in a subsequent paper.

2. We use the conventional notation of the theory of relativity (e. g., summation convention). The derivatives with respect to $\varphi$ we denote by a stroke $\dot{}$, that with respect to $\tau$ or $\lambda^2$ by a comma, viz.:

$$\frac{\partial \varphi}{\partial x^a} = \varphi_{,a}, \quad \frac{\partial \varphi}{\partial \lambda^2} = \lambda \varphi_{,\lambda}, \quad \frac{\partial \varphi}{\partial \tau} = \varphi_{,\tau}, \quad \frac{\partial \varphi}{\partial x^a} = \varphi_{,a}.$$

(a) when written under a letter denotes $\delta_{\alpha a}$.

The gravitational field equations we shall write in the form

$$R_{\alpha\beta} = \kappa (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T),$$

where $R_{\alpha\beta}$ is the Riemann curvature tensor, $T_{\alpha\beta}$ the energy-momentum tensor, $\kappa$ the gravitational constant, and $T$ the energy density.

[439]
where

$$R_{\alpha\beta} = \frac{1}{2} g^{\alpha\gamma} (g_{\gamma\beta,\rho} + g_{\rho\gamma,\beta} - g_{\rho\beta,\gamma} - g_{\gamma\beta,\rho}) + g^{\alpha\lambda} g^{\beta\rho} (T_{\rho\lambda} - T_{\rho\lambda}) T_{\rho\lambda},$$

$T_{\alpha\beta}$ denotes the energy-momentum tensor, $T = T_{\alpha\beta}$, and

$$I_{\alpha\beta} = \frac{1}{2} (g_{\alpha\beta} + g_{\beta\alpha} - g_{\alpha\beta} g_{\alpha\beta}).$$

We assume that in the region of the $x^\sigma$-space, where we look for the field, we can develop $g_{\alpha\beta}$, $g^{\alpha\beta}$ and $T_{\alpha\beta}$ into power series:

$$g_{\alpha\beta} = \sum_{i=0}^{\infty} \delta_{\alpha\beta} i, \quad g^{\alpha\beta} = \sum_{i=0}^{\infty} \delta^{\alpha\beta} i, \quad T_{\alpha\beta} = \sum_{i=0}^{\infty} \delta T_{\alpha\beta} i.$$

From $g_{\alpha\beta} g^{\alpha\beta} = \delta_{\alpha\beta}$ we obtain

$$g^{\alpha\beta} g^{\alpha\beta} = \delta^{\alpha\beta},$$

$$\sum_{i=1}^{\infty} g_{\alpha\beta} g^{\alpha\beta} = 0, \quad n = 1, 2, \ldots$$

From (5) and (6) we can evaluate $g^{\alpha\beta}$ as a function of the $g_{\alpha\beta}, g_{\alpha\beta}, \ldots, g_{\alpha\beta}$ and $g^{\alpha\beta}$ (or vice-versa). The result is

$$g^{\alpha\beta} = \sum_{i=1}^{\infty} (-1)^{i} g^{\alpha\beta} g^{\alpha\beta} g^{\alpha\beta} g^{\alpha\beta} g^{\alpha\beta} g^{\alpha\beta} g^{\alpha\beta} g^{\alpha\beta} g^{\alpha\beta} g^{\alpha\beta}.$$

The sum in (7) is to be taken over all (ordered) sets of positive integers $k_1, k_2, \ldots, k_n$ such that $k_1 + k_2 + \ldots + k_n = l$. Assuming for $T_{\alpha\beta}$ an expansion similar to that of $T_{\alpha\beta}$, we have

$$R_{\alpha\beta} = \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{k_1! k_2! k_3!} \left( g_{\alpha\beta, k_1} + g_{\alpha\beta, k_2} - g_{\alpha\beta, k_3} - g_{\alpha\beta, k_4} \right) +$$

$$+ \frac{1}{k_1! k_2! k_3!} \left( g_{\alpha\beta, k_5} + g_{\alpha\beta, k_6} - g_{\alpha\beta, k_7} - g_{\alpha\beta, k_8} \right) +$$

$$+ \frac{1}{k_1! k_2! k_3!} \left( g_{\alpha\beta, k_9} + g_{\alpha\beta, k_{10}} - g_{\alpha\beta, k_{11}} - g_{\alpha\beta, k_{12}} \right).$$

Equation (8) gives, together with (7), an explicit expression of $R_{\alpha\beta}$ as a function of the $g^{\alpha\beta}$s and their derivatives. The Einstein equations (2) are equivalent to a set of equations:

$$R_{\alpha\beta} = \kappa \left( T_{\alpha\beta} - \frac{1}{2} \sum_{i=1}^{k} g_{\alpha\beta, k} T_{\alpha\beta} \right), \quad l = 0, 1, 2, \ldots$$

3. Let us assume that in our co-ordinate system the tensor $g_{\alpha\beta}$ has the form:

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & f_{\alpha\beta} \end{bmatrix},$$

where $g_{\alpha\beta} = g_{\alpha\beta}(x^\sigma)$, and that the $x^\sigma$-space with the metric $g_{\alpha\beta}$ is (for each $x^\sigma = \text{const.}$) Euclidean. This means that we assume a flat space-time in the zero-order approximation, and use inertial but in general non-cartesian co-ordinate systems.

From these assumptions we have $R_{\alpha\beta} = 0$ ($R_{\alpha\beta} = 0$ follows from the fact that $g_{\alpha\beta}$ is Euclidean), which implies, by virtue of (9, 10, 11), that $T_{\alpha\beta} = 0$. The set of functions $g_{\alpha\beta}$ (or $R_{\alpha\beta}$) $(m, n = 1, 2, 3)$ forms a covariant tensor with respect to transformations of the 3-dimensional space: $x^\sigma \rightarrow x'^\sigma$ (not involving $\lambda$). Likewise $g_{\alpha\beta}$ (or $R_{\alpha\beta}$) is a 3-vector and $g_{\alpha\beta}$ (or $R_{\alpha\beta}$) is a scalar. We shall denote by a semicolon the covariant differentiation in the $x^\sigma$-space with metric $g_{\alpha\beta}$. From (8) we get

$$R_{\alpha\beta} = -\frac{1}{2} g^{\alpha\beta} \left( g_{\mu\nu,\alpha} - g_{\mu\nu,\alpha} + \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha} + R_{\mu\nu} \right) +$$

$$= -\frac{1}{2} g^{\alpha\beta} \left( g_{\mu\nu,\alpha} + R_{\mu\nu} = -\frac{1}{2} g^{\mu\nu} \right) \left( g_{\mu\nu,\alpha} + R_{\mu\nu} \right) +$$

$$+ g^{\alpha\beta} \left( g_{\mu\nu,\alpha} + R_{\mu\nu} \right) \left( g_{\mu\nu,\alpha} + R_{\mu\nu} \right) -$$

$$- g^{\alpha\beta} \left( g_{\mu\nu,\alpha} + R_{\mu\nu} \right) \left( g_{\mu\nu,\alpha} + R_{\mu\nu} \right).$$

The functions $R_{\alpha\beta}$ depend only on $g_{\alpha\beta}, g_{\alpha\beta}, \ldots, g_{\alpha\beta}$ and their derivatives, and can be explicitly evaluated from (8). The right-hand side of (9, $l = 0$) does not involve $g_{\alpha\beta}$, because $T_{\alpha\beta} = 0$. Thus we see that $g_{\alpha\beta}$ enters in (9, $l = 0$) linearly and only through the terms written explicitly in (11)-(13). An essential fact is that the equations (9, $l = 0$) do not involve the time derivatives of $g_{\alpha\beta}$.

Equation (9, $l = 0$) has a unique solution providing the $x^\sigma$-space is pseudo-Euclidean at infinity. This is not, however, the case with equations (9, $l = 0$). If $g_{\alpha\beta}$ is a solution of (9, $l = 0$), then $g_{\alpha\beta}$ defined by the relations

$$g_{\alpha\beta} = g_{\alpha\beta},$$

$$g_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta},$$

$$g_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta},$$

($g_{\alpha\beta}$ - arbitrary functions) is also a solution of these equations. This can be easily seen from (11)-(13). The existence of several solutions of the field equations is in conformity with the general principle of relativity;
the transformation \( g_{\alpha\beta} \rightarrow g'_{\alpha\beta} \) can be regarded as resulting from a co-ordinate transformation [9].

Assuming certain conditions for the co-ordinate system, e. g.,

\[
(15) \quad g^{\alpha\beta}g_{\alpha\beta} = 0, \quad g^{\alpha\beta}(g_{\alpha\gamma\beta} - \frac{1}{2} g_{\alpha\beta\gamma}) = 0 \quad (k = 1, 2, 3),
\]

we can simplify the expression for \( R_{\alpha\beta} \), which becomes, in the case of (15):

\[
(16) \quad R_{\alpha\beta} = -\frac{1}{2} g^{\alpha\beta}g_{\alpha\gamma\beta} + R_{\alpha\beta}^{(0)}
\]

We conclude with a simple remark which may be useful in applying the method.

Let us take a flat \( x \)-space (hence \( T_{\alpha\beta} = 0 \)); this means that there exists a Galilean co-ordinate system in which \( g_{\alpha\beta} \) has the form (1). Let us now introduce a non-inertial co-ordinate system \( x' \), defined by \( \tau = r' \) (i. e. \( x^0 = x'^0 \), \( x^k = x'^k(\tau, x') \)). We have for \( g_{\alpha'\beta'} \):

\[
\begin{align*}
g_{00}' &= g_{00} + 2g_{12}x'_1x'_2 - \frac{1}{2}g_{11}x'^2_1 - \frac{1}{2}g_{22}x'^2_2, \\
g_{01}' &= g_{01} + 2g_{12}x'_1x'_2 - \frac{1}{2}g_{11}x'^2_1 - \frac{1}{2}g_{22}x'^2_2, \\
g_{02}' &= g_{02} + 2g_{12}x'_1x'_2 - \frac{1}{2}g_{11}x'^2_1 - \frac{1}{2}g_{22}x'^2_2.
\end{align*}
\]

Thus, in a general non-inertial co-ordinate system, the metric tensor \( g_{\alpha\beta} \) can be written \( g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + 2g_{12}x_1x_2 \), where \( g_{\alpha\beta}^{(0)} \) has the form (10),

\[
\begin{align*}
g_{\alpha\beta}^{(0)} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
g_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

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REFERENCEs


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THEORETICAL PHYSICS

Solution of One-Body Problem by the Einstein-Infeld Approximation Method

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In the previous paper [1] we presented a generalisation of the "new approximation method". In this paper we shall use that method to evaluate the gravitational field of a point mass, using the notation of [1]. We define \( T_{\alpha\beta} \) so that \( g = h^2 \delta \), where \( k = 6.67 \times 10^{-11} \text{cm}^3 \text{g}^{-1} \text{sec}^{-2} \) (it is perhaps more usual to put \( k = 8\pi G \)). The energy-momentum tensor for a point mass will be represented by an expression involving the three-dimensional Dirac \( \delta \)-function. This method of representing singularities was introduced by Infeld [2].

We assume

\[
\begin{align*}
g_{\alpha\beta} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}, \\
T_0 &= \begin{pmatrix} 1 \end{pmatrix}, \\
T_0 &= \begin{pmatrix} 0 \end{pmatrix}, \quad g_{\alpha\beta} = \begin{pmatrix} 0 \end{pmatrix},
\end{align*}
\]

and denote

\[
\begin{align*}
x^0 &= r, \\
x^1 &= 0, \\
x^2 &= \varphi.
\end{align*}
\]

Further,

\[
\begin{align*}
T_0 &= \frac{1}{c^2} mc^2 \delta(\tau) = mR\delta(\tau) = R T_0, \\
T_0 &= 0 \quad \text{if} \quad \alpha + \beta = 0.
\end{align*}
\]

We also assume that:

(3) the metric \( g_{\alpha\beta} \) is pseudo-Euclidean at infinity;

(4) \( g_{00} = g_{02} = g_{12} = g_{20} = 0 \quad \text{and} \quad g_{01} = g_{11} = g_{22} = 0 \) ("spherical symmetry").

[448]