# EXACT DEGENERATE SOLUTIONS OF EINSTEIN'S EQUATIONS

*by*

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# EXACT DEGENERATE SOLUTIONS OF EINSTEIN'S EQUATIONS*

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## 1. INTRODUCTION

In this lecture we wish to review recent works on gravitational fields that admit a congruence of null geodesics without shear.

Most of our present ideas on waves in general relativity are based on analogies between electromagnetism and gravitation. From the point of view of physics, among the most important features of electromagnetic waves is their ability to transport energy and to carry information. Accordingly, physicists are inclined to consider as gravitational waves those solutions of Einstein's equations *in vacuo* which correspond to a mass changing in time, or contain an arbitrary, information carrying, function. Unfortunately these properties are of such a kind that they do not seem to suggest a method for constructing such solutions. Among electromagnetic waves, there are particularly simple ones, corresponding to a null electromagnetic tensor. With null electromagnetic waves there is associated a remarkably simple and beautiful geometrical structure. Its properties can be stated independently of the electromagnetic field. Thus, one can single out the class of gravitational fields that admit a similar structure and look for waves among them. Gravitational fields belonging to this class are nowadays referred to as 'algebraically special' or 'degenerate'. Before we proceed to review these fields, we wish to describe two typical electromagnetic null fields which have close analogues among our gravitational waves.

Let us consider the following simple situation in special relativity, involving the scattering of an electromagnetic wave on the surface of a perfectly conducting paraboloid of revolution [1]. Let $x = x^1$, $y = x^2$, $z = x^3$ and

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$t = x^4$ be the Cartesian coordinates in flat space-time, $\varrho = \sqrt{x^2+y^2+z^2}$, and $l$ a positive number. Define

$$\tau = t-z-l, \quad \chi = (x+iy)/l,$$
$$\sigma = t-\varrho, \quad \zeta = (x+iy)/(\varrho-z).$$

If $A(\tau, \chi)$ is any complex function analytic in $\chi$, then the real part of[(1)]

$$A(\tau, \chi)\tau,_{[a}\chi,_{b]} \tag{1}$$

represents a solution of Maxwell's equations in empty space. The same can be said of the field

$$A(\sigma, \zeta)\sigma,_{[a}\zeta,_{b]}. \tag{2}$$

Their difference,

$$F_{ab} = A(\tau, \chi)\tau,_{[a}\chi_{b]} - A(\sigma, \zeta)\sigma,_{[a}\zeta,_{b]},$$

satisfies on the surface of the paraboloid, $\sigma = \tau$, the boundary condition

$$F_{[ab}n_{c]} = 0,$$

where $n_c$ denotes a vector orthogonal to the paraboloid. Therefore, the field given by (2) can be interpreted as resulting from reflection, on the conducting paraboloid, of the wave described by (1). If the incident field is simply a *plane* wave, i. e., if $A(\tau,\chi)$ is independent of $\chi$, the resulting field is regular throughout the region $\sigma \leqslant \tau$. The empty space field can be continued into the region $\sigma > \tau$ but this necessarily leads to singularities along the axis of the paraboloid. Both the incident and the reflected field are null. With each of them there is associated a congruence of null geodesics without shear. This is known to be the characteristic property of null electromagnetic fields [2].

The property of a congruence of null geodesics of being shear-free can be described as follows [3]. Think of the null geodesics as of rays of light. Consider a small, plane, opaque object and a plane screen, some distance apart from the object. Suppose that the object and the screen are oriented so that they are orthogonal to the rays of light in their respective rest frames and situated so that the shadow cast by the object can be observed on the screen. The congruence is non-shearing if the shadow, as observed on the screen, is similar in shape to the object.

If $f_{ab}$ is a null solution of Maxwell's equations and $^*f_{ab}$ is its dual,

$$f^{ab}{}_{;b} = 0, \quad {}^*f^{ab}{}_{;b} = 0,$$

then the null vector field $k_a$, defined up to a scalar multiplier by

$$f^{ab}k_b = 0, \quad {}^*f^{ab}k_b = 0$$

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(1) Throughout this lecture the following conventions are used: Latin indices range and sum from 1 to 4. A comma followed by indices denotes ordinary differentiation, a semicolon covariant differentiation; square index-brackets denote antisymmetrization over the indices enclosed.

may be so normalized that

$$k_{a;b}k^b = 0 \tag{3}$$

and

$$(k_{a;b}+k_{b;a})k^{a;b} = (k^a{}_{;a})^2. \tag{4}$$

In this lecture we shall use the term 'ray' to denote a null geodesic belonging to a non-shearing congruence. The trajectories of a vector field $k_a$ subject to (3) and (4) are rays.

## 2. THE LINE-ELEMENT OF DEGENERATE SPACES

From now on we shall confine our attention to empty space-times, i.e., to four-dimensional Riemann spaces of signature $-2$ with vanishing Ricci tensor.

In the theory of gravitation, the analogue of null electromagnetic fields is provided by the class of metrics with degenerate Riemann tensors [3]. The curvature tensor of a non-flat, empty space-time is called algebraically special or degenerate if the equations

$$k_{[a}R_{b]cde}k^c k^d = 0$$

can be satisfied by a vector field $k_a$ which is null and different from zero. We shall also refer to a space or metric as degenerate if its Riemann tensor has this property. Sachs has shown that if such a $k_a$ exists, it must be tangent to a congruence of rays. Conversely, if an empty space-time admits a null congruence of this kind, its metric is degenerate [4], [5].

One is thus led to consider a four-dimensional, normal hyperbolic Riemann space $V_4$ that admits a null vector field $k_a$ satisfying equations (3) and (4). The curves $x^a = x^a(\varrho)$ defined by

$$\frac{dx^a}{d\varrho} = k^a$$

are null geodesics. Let us introduce coordinates in $V_4$ such that $x^3$ coincides with the affine parameter $\varrho$ and the null geodesics are coordinate lines of $x^3$ (i.e. $x^1=\xi$, $x^2=\eta$ and $x^4=\sigma$ are constant along the rays). With this choice of coordinates, $k^a=\delta^a_3$ and $g_{33}=0$. It follows from the geodetic condition, Eq. (3), that the covariant components, $k_a=g_{a3}$, are independent of $\varrho$. Let us introduce the following vectors,

$$l_a = \varrho_{,a} + \frac{1}{2}ck_a,$$

$$x^1_a = P(\xi_{,a} - ak_a),$$

$$x^2_a = P(\eta_{,a} - bk_a),$$

and choose $a$, $b$ and $c$ so that

$$l_a l^a = 0, \quad x^\lambda_a l^a = 0, \qquad \lambda = 1,2.$$

The metric tensor can then be written in the form

$$g_{ab} = k_a l_b + k_b l_a + \gamma_{\varkappa\lambda} x^\varkappa_a x^\lambda_b,$$

where $\gamma_{\varkappa\lambda}$ is a symmetric, two by two matrix. If one chooses $P$ so that det $\gamma_{\varkappa\lambda}=1$, the non-shearing condition, Eq. (4), reduces to

$$\partial\gamma_{\varkappa\lambda}/\partial\varrho = 0.$$

By a coordinate transformation one can impose the further restriction

$$\gamma_{\varkappa\lambda} = -\delta_{\varkappa\lambda}.$$

If we denote $k_a dx^a$ by $d\Sigma$ (not a perfect differential, in general), the line-element can be written as

$$ds^2 = -P^2[(d\xi - a d\Sigma)^2 + (d\eta - b d\Sigma)^2] + 2d\varrho d\Sigma + c d\Sigma^2, \tag{5}$$

where $a$, $b$, $c$ and $P$ are functions of all the coordinates and the components of $k_a$ are independent of $\varrho$.

## 3, SPACES WITH CURLING RAY

The quantity $\omega$ defined by

$$\omega^2 = \frac{1}{2} k_{[a;b]} k^{a;b}$$

measures the amount of rotation of rays. It vanishes for a hypersurface-orthogonal congruence of rays.

The field equation

$$R_{ab} k^a k^b = 0$$

reduces for the metric (5) to

$$P^{-1}\partial^2 P/\partial\varrho^2 = \omega^2. \tag{6}$$

The case of $\omega \neq 0$ has been recently investigated by Newman, Tamburino and Unti [6]. Their paper, based on a very elegant technique developed by Newman and Penrose [7], containes an interesting class of new exact solutions in closed form. All their metrics are of degenerate type I, have curling rays and constitute a generalization of the Schwarzschild solution. Newman, Tamburino and Unti claim that all algebraically special metrics with $\omega \neq 0$ are of degenerate type I. This would be a significant result, for in the linearized gravitational theory type II null solutions can be easily constructed [8]. By an appropriate choice of coordinates, the components of the Newman-Tamburino-Unti metrics can be reduced to

$$P = p^{-1}\sqrt{\varrho^2 + \alpha^2}; \quad a = b = 0; \quad c = K - \frac{2m\varrho + 2\alpha^2 K}{\varrho^2 + \alpha^2}$$

$$k_a = (\alpha\eta p^{-1}, \ -\alpha\xi p^{-1}, \ 0, \ 1);$$

where

$$p = 1 + \tfrac{1}{4}K(\xi^2+\eta^2); \quad K = -1,\ 0 \text{ or } 1;$$

$m$ and $\alpha \neq 0$ are constants.

As all these metrics are stationary, they are not interesting from the point of view of gravitational radiation theory.

## 4. SPACES WITH NON-EXPANDING RAYS

The class of known gravitational fields with non-rotating rays is much larger than that corresponding to solutions with $\omega \neq 0$. Many of these fields possess properties characteristic for waves. It should be noted that both the null electromagnetic fields described at the beginning of this lecture are connected with non-rotating rays.

If we assume that $k_a$ is hypersurface-orthogonal, coordinates in $V_4$ can be chosen so that $k_a = \sigma_{,a}$ and $d\Sigma$ becomes simply $d\sigma$. The two-dimensional surfaces $\sigma$=const., $\varrho$=const. can be interpreted as wavefronts. For $\omega$=0 equation (6) leads to

$$P = Q\varrho + R,$$

where $Q$ and $R$ are independent of $\varrho$. As

$$k^a{}_{;a} = 2P^{-1}\partial P/\partial\varrho$$

one has to consider two cases depending on whether or not $Q$ vanishes.

Let us first take the case of non-expanding rays,

$$k^a{}_{;a} = 0, \quad \partial P/\partial\varrho = 0.$$

Sachs [3] has shown that the Riemann tensor for an empty space-time with a non-rotating, non-expanding family of rays has the form

$$R_{abcd} = \mathrm{II}_{abcd} + \varrho \mathrm{III}_{abcd} + \varrho^2 N_{abcd} \tag{7}$$

where $\mathrm{II}_{abcd}$ $\mathrm{III}_{abcd}$ and $N_{abcd}$ denote, respectively, tensors of Petrov's type II, III and II null, or more special tensors. They are covariantly constant along the rays.

The Einstein field equations have not yet been solved for the general case of a space endowed with non-expanding rays. However, many particular solutions with rays of this type have been known since a long time.

Brinkmann as early as in 1923 [9] had found a class of metrics which were later rediscovered[2] and described as plane-fronted gravitational

[2] In fact, they were independently discovered by I. Robinson in 1956, by J. Hely (*C. R. Acad. Sci., Paris* **249**, 1867 (1959)) and by A. Peres (*Phys. Rev. Letters* **3**, 571 (1959)). The physical meaning of these metrics and their connection with plane gravitational waves were recognized for the first time by I. Robinson (unpublished, presented at several seminars in England and in Warsaw in 1958)—*the Editors*.

waves with parallel rays [10]. They can be characterized by the statement that the corresponding propagation vector $k_a$ is covariantly constant. The Brinkmann waves are of Petrov's type II null.

Plane gravitational waves, discovered by Einstein and Rosen [11], disregarded by them on the ground that they possessed singularities, restored to good standing by Bondi, Pirani and Robinson [12], constitute a subclass of the Brinkmann metrics.

A more general class of exact solutions has been recently found by several authors [13]. The metrics are characterized by the existence of a 'recurrent' propagation vector:

$$k_{a;b} = \frac{1}{2} G k_a k_b.$$

By using the field equations, $R_{ab} = 0$, their line-element can be brought to the form

$$ds^2 = -|d\zeta - f d\sigma|^2 + 2 d\varrho d\sigma + (G\varrho - 2H) d\sigma^2,$$

where $f(\zeta, \sigma) = u + iv$ is an analytic function of $\zeta = \xi + i\eta$, $G = \dfrac{\partial u}{\partial \xi} + \dfrac{\partial v}{\partial \eta}$ and $H$ is a function of $\xi, \eta$ and $\sigma$ which may be determined from the equation

$$\frac{\partial^2 H}{\partial \xi^2} + \frac{\partial^2 H}{\partial \eta^2} + G^2 + \frac{\partial G}{\partial \sigma} + u\frac{\partial G}{\partial \xi} + v\frac{\partial G}{\partial \eta} = 0.$$

The Riemann tensor of these spaces is given by

$$R_{abcd} = \mathrm{III}_{abcd} + \varrho N_{abcd},$$

where the symbols have the same meaning as in Eq. (7).

Kundt [10] exhibited all the space-time metrics with non-expanding rays which are of type III or II null. They all have plane wave-fronts and can be characterized by this property.

## 5. SPACES WITH EXPANDING HYPERSURFACE-ORTHOGONAL RAYS

In the case of diverging rays, $k^a{}_{;a} \neq 0$, one can get

$$k^a{}_{;a} = 2/\varrho$$

by a coordinate transformation of the form $\varrho \to \varrho + \varphi(\xi, \eta, \sigma)$. $P$ can then be written as $p^{-1}\varrho$, where $p$ is a function of $\xi$, $\eta$ and $\sigma$ only. The field equations and the remaining freedom of coordinate transformations can be used to reduce the line-element to [14]

$$ds^2 = -\varrho^2 p^{-2}(d\xi^2 + d\eta^2) + 2 d\varrho d\sigma + (-2m\varrho^{-1} + K - 2H\varrho) d\sigma^2, \qquad (8)$$

where

$$m = m(\sigma), \quad K = \Delta \ln p, \quad H = \frac{\partial}{\partial\sigma}\ln p,$$

$$\Delta \equiv p^2\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right),$$

and $p = p(\xi, \eta, \sigma)$ is subject to the condition

$$4\left(\frac{\partial}{\partial\sigma} - 3H\right)m - \Delta K = 0. \tag{9}$$

The curvature tensor is given now by

$$R_{abcd} = \varrho^{-1}N_{abcd} + \varrho^{-2}\mathrm{III}_{abcd} + \varrho^{-3}D_{abcd},$$

where $D_{abcd}$ denotes a tensor of Petrov's type I degenerate.

For $m = 0$, Eq. (9) reduces to $\Delta K = 0$ and the $\sigma$-dependence of $p$ is arbitrary. Solutions with $m = 0$ are of type III, II null or flat. Moreover, if $K$ is independent of $\xi$ and $\eta$ it can be reduced to $-1$, 0, or 1 by a change of $\sigma$ into a function of $\sigma$. In all three cases the Riemann tensor is of type II null, $R_{abcd} = \varrho^{-1}N_{abcd}$. For $K = 1$ the wave-fronts $\sigma$ = const., $\varrho$ = const. are spheres of radius $\varrho$ and the corresponding waves resemble the spherical null electromagnetic waves described at the beginning of this lecture. As usually waves are named after the geometry of their wave-fronts, the designation 'spherical gravitational waves' seems appropriate for the non-flat metrics (8) with $m = 0$ and $K = 1$. Spherical gravitational waves suffer from singularities, similar in nature to the line singularity that appears in null spherical electromagnetic waves, Eq. (2), when one shrinks the paraboloid by letting $l$ tend to zero.

For $m \neq 0$ one can normalize $\sigma$ so as to have $m = 1$. The solution is then completely specified by giving $p$ as a function of $\xi$ and $\eta$ for a definite value of $\sigma$, say 0. Indeed, the parabolic differential equation (9) enables us then to calculate $p$ for other values of $\sigma$. The function $p$ defines a one-parameter family of two-dimensional surfaces, $V_2(\sigma)$, with the line-element $p^{-2}(d\xi^2 + d\eta^2)$. If $V_2(0)$ is of constant curvature, $\partial p/\partial\sigma = 0$, the line-element (8) is static and the Riemann tensor is of type I degenerate and falls off as $1/\varrho^3$. In the general case, the curvature tensor is of type II and contains the $1/\varrho$ term typical for waves. Choose now for $V_2(0)$ a regular, closed surface of variable curvature, e.g., the surface of an ellipsoid. At least for a finite neighbourhood of $\sigma = 0$, Eq. (9) defines a family of regular, closed surfaces $V_2(\sigma)$. The corresponding $V_4$, with $ds^2$ given by (8), has only a point singularity at the origin $\varrho = 0$. This shows that one can construct a nearly spherical gravitational wave which, at least for a finite range of $\sigma$ is regular everywhere except the origin. A point singularity is usually interpreted as representing the source of radiation.

The authors are indebted to R. Penrose for the argument of the last paragraph.

## REFERENCES

[1] H. Bateman, The mathematical analysis of electrical and optical wave-motion, Dover Publications, 1955.

[2] I. Robinson, *J. Math. Phys.* **2**, 290 (1961).

[3] R. Sachs, *Proc. Roy. Soc.* **A264**, 309 (1961).: article in Recent Developments in General Relativity, Pergamon Press and PWN 1962.

[4] J. N. Goldberg and R. Sachs, A theorem on Petrov's types, ASTIA *preprint*, London 1961.

[5] I. Robinson and A. Schild, seminar delivered at the Conference on Relativistic Theories of Gravitation, Warszawa and Jabłonna 1962, in this volume.

[6] E. Newman, L. Tamburino and T. Unti, Curling geodesic rays with vanishing shear, *preprint*, 1962.

[7] E. Newman and R. Penrose, *J. Math. Phys.* **3**, 566 (1962).

[8] A. Trautman, *Proc. Roy. Soc.* **A270**, 326 (1962).

[9] H. W. Brinkmann, *Proc. Nat. Acad. Sci., Wash.* **9**, 1 (1923).

[10] W. Kundt, *Z. f. Phys.* **163**, 77 (1961).

[11] A. Einstein and N. Rosen, *J. Franklin Inst.* **223**, 43 (1937); N. Rosen, *Phys. Z. Sovjet.* **12**, 366 (1937).

[12] H. Bondi, *Nature* **179**, 1072 (1957); H. Bondi, F. A. E. Pirani and I. Robinson, *Proc. Roy. Soc.* **A251**, 519 (1959).

[13] R. Debever and M. Cahen, *C. R. Acad. Sci. Paris* **251**, 1160 and 1352 (1960); J. Goldberg and P. Kerr, *J. Math. Phys.* **2**, 332 (1961).

[14] I. Robinson and A. Trautman, *Phys. Rev. Letters* **4**, 431 (1960); *Proc. Roy. Soc.* A**265**, 463 (1962).