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# ON THE PROPAGATION OF INFORMATION BY WAVES

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## I. INTRODUCTION

Gravitational waves are usually defined by their geometric properties. For example, plane waves are invariant under a 5-parameter group of motions [1]. Pirani's definition of pure radiation fields refers to the algebraic structure of the Riemannian tensor [2]. However, the elegant geometrical approach cannot easily be applied to the problem of spherical gravitational waves. In this case no more than axial symmetry can be assumed. The Riemannian tensor of a spherical wave is expected to be of type I.

There is another important property of waves, both linear and gravitational: waves can propagate information. This means that wave-like solutions depend on arbitrary functions, the shape of which contains the information carried by the wave. This is obviously true of both linear waves and the known gravitational waves. The dependence on these arbitrary functions can be very simple, as in the case of plane waves, or more complicated as in the case of cylindrical waves.

We shall show how one can deduce certain properties of waves from the fact that they carry information. We shall begin with very simple remarks on linear waves. In the gravitational case we have only been able to rederive Robinson's line-element for plane-fronted waves [3], [4]. However, it may be possible to obtain some information about spherical waves by the method outlined in this note.

## 2. WAVES IN LINEAR THEORIES

Let us start with scalar waves in Minkowski space. We shall consider solutions of the wave equation

$$(g^{\mu\nu}\partial_{\mu}\partial_{\nu}+\varkappa^{2})\varphi=0, \quad g^{00}=1, \quad g^{0k}=0, \quad g^{ik}=-\delta^{ik}, \quad (1)$$

which have the form

$$\varphi = \varphi(x, F(\sigma)) \tag{2}$$

where  $F(\sigma)$  is an arbitrary (twice differentiable) function of the scalar  $\sigma = \sigma(x)$ . The form (2) includes plane and spherical but not cylindrical waves. Derivatives with respect to F will be denoted by primes. A comma followed by an index will denote differentiation with respect to the x's which appear in  $\varphi$  explicitly; for instance:

$$\partial_{\nu}\varphi = \varphi_{,\nu} + \varphi' F' \sigma_{,\nu}, \qquad \sigma_{,\nu} \equiv \partial_{\nu}\sigma.$$

The function F being arbitrary, we obtain from (1):

$$g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu}=0\,, \qquad (3a)$$

$$g^{\mu\nu}(2\varphi'_{,\mu}\sigma_{,\nu}+\varphi'\sigma_{,\mu\nu})=0,$$
 (3b)

$$g^{\mu\nu}\varphi_{,\mu\nu}+\varkappa^2\varphi=0. \qquad (3c)$$

The surfaces  $\sigma = \text{const}$  turn out to be null. Equation (3b) is a typical propagation equation (cf., for example, [5]) and the meaning of (3c) is obvious.

It is easy to write down equations analogous to (3) in Maxwell's theory. In order to include both plane and some spherical waves let us take for the electromagnetic tensor:

$$f_{\mu\nu} = {}^{0}f_{\mu\nu}(x)F(\sigma) + {}^{1}f_{\mu\nu}(x)F'(\sigma) + {}^{2}f_{\mu\nu}(x)F''(\sigma). \tag{4}$$

Maxwell's equations give:

$$^2f^{\mu\nu}\sigma_{,\nu}=0, \qquad (5a)$$

$${}^{1}f^{\mu\nu}\sigma_{,\nu} + {}^{2}f^{\mu\nu}_{,\nu} = 0, \tag{5b}$$

$${}^{0}f^{\mu\nu}\sigma_{,\nu} + {}^{1}f^{\mu\nu}_{,\nu} = 0, \qquad (5c)$$

$$^0f^{\mu\nu}_{\phantom{\mu\nu}}=0, \qquad (5d)$$

and similar equations with the f's replaced by their duals. If follows that  $\sigma_{,\nu}$  is null and  ${}^2f$  is a null bivector which satisfies a propagation equation identical in form to (3b). This means  ${}^2fF''$  is the pure radiation part of the wave; if  $\sigma_{,\nu}$  satisfies Robinson's equation [3] and  $g^{\mu\nu}\sigma_{,\mu\nu} \neq 0$  then  ${}^2f$  falls off like 1/r [5]. The propagation equations for  ${}^0f$  and  ${}^1f$  are

$$2\operatorname{Prop}{}^{0}f+{}^{0}f\square\sigma+\square^{1}f=0,$$

$$2\operatorname{Prop}^{1}f+f\Box\sigma+\Box^{2}f=0.$$

(We have suppressed the indices and introduced the operators  $\operatorname{Prop} = g^{\mu\nu}\sigma_{,\mu}\partial_{,\mu}$  [3],  $\square = g^{\mu\nu}\partial_{\mu}\partial_{\nu}$ .)

We can easily write down the electromagnetic potentials corresponding to the field given by q. (4). The potentials of a typical spherical wave produced by a Hertz dipole are of the form:

$$A_{\mu} = {}^{0}A_{\mu}(x)F(\sigma) + {}^{1}A(x)F'(\sigma)\sigma_{,\mu} + {}^{2}A(x)F''(\sigma)\sigma_{,\mu}. \tag{6}$$

In this case we have [5]:

$$\sigma_{\varrho}^{1}f^{\varrho}_{[\mu}\sigma_{,\nu]}=0$$

and Robinson's equation follows from Eqs (5). For the potentials of a plane wave depending on one arbitrary function of  $\sigma$  we can choose the expression

$$A_{\mu} = A(x)F(\sigma)\sigma_{,\mu}. \tag{7}$$

Electromagnetic potentials are not interesting from the point of view of the theory but their form may suggest possible metrics corresponding to gravitational waves.

#### 3. GRAVITATIONAL WAVES

It is not quite easy to apply our method to the gravitational field. Arbitrary functions can be introduced into every metric by coordinate transformations. Metrics containing only arbitrary functions which can be removed by a change of coordinates do not carry information and cannot be called waves. We do not know, however, how to distinguish beforehand between spurious and genuine arbitrary functions. We are therefore forced to check the character of a metric only after the field equations have been solved.

The second difficulty is of a technical character but is by no means less serious. Suppose we take for  $g_{\mu\nu}$  a simple function of F, say a polynomial in F and its derivatives up to a certain order. However, the corresponding inverse tensor  $g^{\mu\nu}$  need not be simple; in general it will be an infinite series in F and its derivatives. The field equations will split into an infinite and overdetermined set of equations and the whole procedure will closely resemble an approximation method.

In order to avoid this difficulty we shall investigate only the simplest case, when both  $g_{\mu\nu}$  and  $g^{\mu\nu}$  depend linearly on F alone. We can show that the metric must then be of the form:

$$g_{\mu\nu} = g_{\mu\nu}(x) + F(\sigma)h_{\mu}(x)h_{\nu}(x)$$
 (8)

where  $h_{\nu}$  is a null vector with respect to the background metric  $g_{\mu\nu}$ :

$$g^{\mu\nu}h_{\mu}h_{\nu} = 0, \qquad g^{\mu\varrho}g_{\varrho\nu} = \delta^{\mu}_{\nu}. \tag{9}$$

The metric tensor with upper indices has the form:

$$g^{\mu\nu} = g^{\mu\nu} - Fh^{\mu}h^{\nu}$$

where

$$h^{\mu}=g^{\mu\nu}h_{\nu}.$$

In the following we shall use  $g^{\mu\nu}$  to raise the indices. Covariant differentiation with respect to the background metric will be denoted by a semicolon. The operators Prop and  $\square$  will also be understood to be taken with respect to  $g_{\mu\nu}$ .

Einstein's field equations  $R_{\mu\nu}=0$  lead to the following conditions on  $g_{\mu\nu},\ h_\varrho$  and  $\sigma$ :

- (a) the Riemann space defined by the background metric must by empty,  $R_{\mu\nu}=0$ ;
  - (b)  $g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu}=0;$
  - (c)  $h^{\nu}\sigma_{,\nu}=0$ , therefore  $h_{\nu}=h\sigma_{,\nu}$ ;
  - (d)  $\Box \sigma = 0;$
  - (e)  $(\text{Prop})^2 H = 0$ , where  $H = h^2$ ;
  - (f)  $2(\operatorname{Prop} H_{,(\alpha)}\sigma_{,\beta}) (\operatorname{Prop} H)_{,(\alpha}\sigma_{,\beta)} + \operatorname{Prop}(H\sigma_{;\alpha\beta}) \frac{1}{2}\Box H\sigma_{,\alpha}\sigma_{,\beta} = 0$ .

If we choose the flat metric for  $g_{\mu\nu}$ , we can take

$$\sigma_{;\alpha\beta}=0, \qquad \sigma_{,\alpha}\neq 0, \qquad x^0\equiv \sigma$$

and  $x^1$  as the preferred parameter along rays. The general solution of (e) and (f) is of the form:

$$H = M(x^0, x^2, x^3) + x^1 N(x^0)$$

with M satisfying

$$M_{.22} + M_{.33} = 0.$$
 (10)

The arbitrary function N is spurious, i.e., it can be transformed away. The line-element reduces to that of plane-fronted waves,

$$ds^2 = M(x^0, x^2, x^3)(dx^0)^2 + 2 dx^0 dx^1 - (dx^2)^2 - (dx^3)^2$$

with M restricted by equation (10) only and  $F(x^0)$  absorbed into M. Spherical gravitational waves will be more complicated than plane waves in several respects. First of all, we cannot expect both

 $g_{\mu\nu}$  and  $g^{\mu\nu}$  to be simple functions of F. Secondly, the simplest gravitational spherical wave may be described as a "quadrupole" wave. Therefore, it will probably be necessary to introduce into the metric derivatives of F of a higher order than in equation (4).

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