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Analytic solutions of Lorentz-invariant linear equations

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All the algebraically special, wave-like solutions of Einstein's equations so far discovered admit hypersurface-orthogonal propagation vectors. Little is known about metrics with curling propagation vectors. Even in electrodynamics, few solutions of this type have been exhibited.

This note presents a method of constructing classes of new solutions to linear, special relativistic partial differential equations. In particular, the method may be used to produce null, curling solutions of Maxwell's and linearized Einstein's equations. It consists in a generalization of a procedure used by Synge to obtain regular wave-packets from the fundamental solution $(t^2-x^2-y^2-z^2)^{-1}$ (Synge 1960*a,b*).

The method may be explained by the example of a scalar wave equation,

$$g^{kl}\partial^2\phi/\partial x^k\partial x^l = 0 \quad (k, l = 0, 1, 2, 3),$$

where g^{kl} is the Minkowski metric tensor in Cartesian co-ordinates. Let $\phi(x^k)$ be a solution of the wave equation analytic in all the four co-ordinates $(t, x, y, z) = x^k$. The function ϕ can be continued for complex values of the co-ordinates. The resulting function, $\phi = \phi(z^k)$, $z^k = x^k + iy^k$, satisfies the equation

$$g^{kl}\partial^2\phi/\partial z^k\partial z^l = 0. \tag{1}$$

Let $z^k \to z'^k$ be a complex inhomogeneous Lorentz transformation,

$$z^k = L^k_l z'^l + a^k, \quad g_{kl}L^k_m L^l_n = g_{mn}, \quad g_{km}g^{ml} = \delta^l_k.$$

The transformed field, $\qquad \phi'(z^k) = \phi(L^k_l z^l + a^k),$

is also a solution of equation (1). Therefore, $\phi'(x^k)$ is a (complex) solution of the equation, defined over the usual real Minkowski space.

A similar procedure can be applied to fields satisfying Maxwell's equations or the linearized gravitational equations. By analytic continuation, a null Maxwell field goes into a null field and the Petrov type of the linearized Riemann tensor is preserved. With a null electromagnetic field there is associated a congruence of null rays which are geodetic and shear-free (Robinson 1961). The field obtained by analytic continuation from a null field also admits a congruence of null, non-shearing geodesics but the coefficients of rotation of these two congruences need not be the same. Similar remarks apply to algebraically special linearized gravitational fields.

Let f_{kl} denote the electromagnetic tensor, $^*f_{kl}$ its dual and $F_{kl} = f_{kl} + \mathrm{i}^*f_{kl}$. Plane-fronted electromagnetic waves may be written in the form

$$F_{kl} = A(\sigma, \zeta)\,\sigma_{,[k}\zeta_{,l]}, \tag{2}$$

where $$\sigma = k_l x^l, \quad \zeta = m_l x^l, \tag{3}$$

k_l, m_l are two complex, constant, orthogonal null vectors and A is a function analytic in ζ. Plane-fronted waves with amplitudes A analytic in both variables are carried by complex Lorentz transformations into waves of the same kind.

Choose k_i to be real and let l_i be another real null vector such that $k_i l^i = 1$, $l_i m^i = 0$. The conformal point transformation

$$x^i \to (l_b x^b)^{-1}(k^i + x^i - l^i k_a x^a - k^i l_a x^a - \tfrac{1}{2}l^i x_a x^a),$$

applied to F_{kl} given by equations (2) and (3) leads to a null electromagnetic field with diverging, non-rotating rays. The electromagnetic tensor is of the form (2) with

$$\sigma = -\tfrac{1}{2}x_a x^a/(l_b x^b), \quad \zeta = m_a x^a/l_b x^b.$$

Assuming that A is analytic in both variables, one can perform the complex translation,

$$x^a \to x'^a = x^a + \mathrm{i}\lambda k^a, \quad \lambda = \text{const}.$$

Replacing in formula (2) the scalars σ and ζ by

$$\sigma'(x^a) = \sigma(x^a - \mathrm{i}\lambda k^a), \quad \zeta'(x^a) = \zeta(x^a - \mathrm{i}\lambda k^a),$$

one obtains a null field with a family of diverging, rotating rays tangent to the vector

$$\sigma'_{,\iota} - 2i\lambda\bar{\zeta}'\zeta'_{,\iota}$$

(Robinson 1961, private communication).

Null spherical hypersurface-orthogonal electromagnetic waves are of the form (2) with σ and ζ defined through

$$\sigma + \rho = t, \quad \rho = (x^2 + y^2 + z^2)^{\frac{1}{2}},$$

$$\rho\zeta/(1 + \tfrac{1}{4}\zeta\bar{\zeta}) = x + iy.$$

A complex translation of the form

$$t' = t, \quad x' = x, \quad y' = y, \quad z' = z + i\lambda,$$

leads also to a null field with a curling propagation vector.

It is easy to write down analogous solutions in the linearized gravitational theory.

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