BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 10, 1958

THEORETICAL PHYSICS

On Gravitational Radiation Damping

by

A. TRAUTMAN

Presented by L. INFELD on July 28, 1958

1. The problem of gravitational radiation has several aspects. In this paper those which are related to the free motion of gravitating bodies are considered.

In Newtonian mechanics the initial positions and velocities of point-masses determine their motion completely. The situation is different in electrodynamics where the initial values of the field are required besides information concerning changes. Two *free* point-charges of opposite signs may move uniformly around a circle in a standing-wave electromagnetic field. However, the same charges may alternatively produce outgoing radiation. Their motion will not then be periodic; they will undergo damping. Which of these cases occurs in any particular system depends on the initial and boundary conditions.

The question arises as to whether the situation in General Relativity resembles that in Newtonian mechanics rather than that in electrodynamics. As yet, the most powerful tool for dealing with the problem of motion is the Einstein-Infeld-Hoffmann "new" approximation method [1]. This provides a description of a set of motions which are compatible with the field equations and which, in the first significant order, correspond to the Newtonian motions. The original EIH method is based on certain assumptions corresponding to the choice of symmetric potentials in electrodynamics. Uniform circular motion of two bodies of equal mass is possible within the original framework of the new approximation method, but "damped" motion along a spiral path is not. The problem of whether the EIH method discloses *all* possible motions of bodies interacting through the gravitational field has been the object of several papers. Infeld [2] supplemented the original EIH expansions by certain "radiation terms" and showed that they do not contribute to the equations of motion of the 7th order (the Newtonian equations being of the 4th). Hu [3] performed the next step of the approximation procedure and found the radiative corrections to the equations of motion in the 9th order. The result of his lengthy calculations was paradoxical: it turned out that the total energy of a radiating double-star system would increase. Infeld and Scheidegger [4] investigated the possibility of eliminating the radiation terms by means of co-ordinate transformations. Later, Goldberg [5] found radiative solutions which cannot be annihilated in this way.

The aim of the present paper is to show that there are free gravitational motions which differ from those described by the original formulation of the EIH method. As an illustration, the calculations referring to the problem of two bodies of equal mass are performed by a method slightly different from that of Hu. On the assumption that the Newtonian motion is circular, the equations of motion up to the 9th order turn out to be of the type $\ddot{x}+2\alpha\dot{x}+\omega^2(x)x=0$, where $\alpha=\text{const}>0$.

2. The EIH method is based on a special approximation procedure and provides a way of obtaining the equations of motion. The fact that the equations of motion follow from the field equations is due to the general covariance of Einstein's theory, but the new approximation method can also be applied in other field theories.

Let $\varphi=\varphi(x^\nu, c)$ denote a function of co-ordinates x^ν $(\nu=0,1,2,3)$ depending on a parameter c. In the new approximation method a new time variable $t=x^0/c$ is introduced and it is assumed that φ can be expanded in a power series in $1/c$.

$$\varphi=\varphi(ct, x^k, c)=\sum_{m=0}^{\infty} c^{-m}\underset{m}{\varphi}(t, x^k), \qquad (k=1,2,3). \tag{1}$$

It follows that $\partial\underset{m}{\varphi}/\partial x^0=c^{-1}\partial\underset{m}{\varphi}/\partial t=c^{-1}\dot{\underset{m}{\varphi}}$ is of order $m+1$. We choose $c=1$, bearing in mind subsequently that differentiation with respect to time increases the order of a term by one.

If φ satisfies the wave equation $\Box\varphi=0$, then the fields $\underset{m}{\varphi}$ are subject to the relations

$$\Delta\underset{0}{\varphi}=0, \quad \Delta\underset{1}{\varphi}=0, \quad \Delta\underset{2}{\varphi}=\ddot{\underset{0}{\varphi}}, \ldots, \quad \Delta\underset{m}{\varphi}=\ddot{\underset{m-2}{\varphi}}, \ldots, \tag{2}$$

The retarded spherical wave

$$\varphi=a(t-r)/r=\sum_{m=0}^{\infty}(-1)^m(m!)^{-1}r^{1-m}d^m a(t)/dt^m \tag{3}$$

is a solution of (2). By omitting the odd terms in (3) one obtains the half-advanced, half-retarded wave $\frac{1}{2}[a(t-r)+a(t+r)]$. In order to make easier the comparison with the original works on the EIH method $a(t)$ is assumed to be of second order. The odd terms in (3) are called "radiation terms", the first of which, $\underset{3}{\varphi} = -\dot{a}(t)$, is a function of time alone. It is important to note that $\underset{m}{\varphi} = 0(r^{m-3})$ for large r, this being a general property of solutions of the inhomogeneous wave equation with a spatially bounded source. If the field φ is periodic and if λ denotes the corresponding wave-length, then the ratio of two consecutive terms in (1) is of order r/λ (where r is the distance from the source). The first few terms of the series (1) will closely approximate the field φ only in the region $r \ll \lambda$; the EIH method is not well-suited to the investigation of the field in the wave zone.

In *electrodynamics* the situation is quite similar, and we shall only stress certain peculiarities due to gauge-invariance. If A^a is a potential satisfying the Lorentz condition $A^a{}_{,a} = 0$, then Maxwell's equations read

$$\Box A^a = -4\pi j^a, \quad j^a{}_{,a} = 0 . \tag{4}$$

It is convenient to assume that j^0 and j^k are of the second and third orders, respectively. For regular and spatially bounded j^a we may solve (4) using, e. g., the retarded Green function. On expanding the solution into an EIH power series we find that the first radiation term

$$\underset{3}{A^0} = -\int j^0{}_{,0} dV$$

vanishes because of the conservation of charge: $j^0{}_{,0} + j^k{}_{,k} = 0$. Thus the first non-vanishing radiation term is of the 4th order, $\underset{4}{A^k}$. For large , and for $m \geqslant 3$ we have $\underset{m}{A^a} = 0(r^{m-4})$.

An analogous discussion is possible in the *linearized theory of gravitation.* If we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (with $h_{\mu\nu}$ small), $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\varrho\sigma}h_{\varrho\sigma}$ and assume $\gamma^{\mu\nu}{}_{,\nu} = 0$, then Einstein's equations become

$$\Box \gamma^{\mu\nu} = 16\pi T^{\mu\nu}, \quad T^{\mu\nu}{}_{,\nu} = 0 , \tag{5}$$

where terms non-linear in $h_{\alpha\beta}$ have been neglected. The functions T^{00}, T^{0k} and T^{ik} can be assumed to be of second, third and fourth orders, respectively. The $\underset{3}{\gamma_{00}}$ and $\underset{4}{\gamma_{0k}}$ vanish because of the conservation laws [2], hence $\underset{5}{\gamma_{00}}, \underset{5}{\gamma_{ik}}$ are the first non-vanishing radiation terms.

Taking into account the conservation laws one may easily show that, for large r and for $m \geqslant 4$, $\underset{m}{\gamma_{\mu\nu}}$ behaves like r^{m-5}.

3. The structure of *Einstein's equations* is such that we can, if we wish, choose solutions of the form

$$\begin{aligned} g_{00} &= 1 + \underset{2}{g_{00}} + \underset{4}{g_{00}} + \underset{6}{g_{00}} + \dots \\ g_{0k} &= \underset{3}{g_{0k}} + \underset{5}{g_{0k}} + \underset{7}{g_{0k}} + \dots \\ g_{ik} &= -\delta_{ik} + \underset{2}{g_{ik}} + \underset{4}{g_{ik}} + \underset{6}{g_{ik}} + \dots . \end{aligned} \tag{6}$$

By analogy with the scalar wave equation and Maxwell's theory, solutions of the form (6) may be interpreted as representing standing-wave fields. In order to get solutions corresponding to "retarded" or "advanced" fields the series (6) must be supplemented with the missing *radiation terms*, i. e. those odd in g_{00} and g_{ik} and even in g_{0k}.

A solution for $\underset{n}{g_{\mu\nu}}$ $(n \geqslant 4)$ contains, in general, both terms linear and non-linear in the masses. The linear part of $\underset{n}{g_{\mu\nu}}$ is easily calculated from the linearized theory by solving (5). We may expect $\underset{n}{g_{\mu\nu}}$ also to behave like r^{n-5} $(n \geqslant 4)$, unless some of the non-linear terms in $\underset{n}{g_{\mu\nu}}$ cancel with the r^{n-5} terms in the linear part. In general, one cannot impose on the expanded metric the condition $\lim\limits_{r\to\infty} \underset{n}{g_{\mu\nu}} = 0$. However, this does not necessarily mean that the metric is non-flat at infinity.

The first radiation terms satisfy linear homogeneous equations and we may expect them to be linear in the masses so that their form can be derived from the linearized theory. If we take $T^{\mu\nu}$ in the form proposed by Infeld [6] for spherically symmetric non-rotating bodies

$$\sqrt{-g}\,T^{\mu\nu} = \sum m\dot{\xi}^{\mu}\dot{\xi}^{\nu}\delta_{(3)}\big(x^s - \xi^s(t)\big)\,dt/ds\,, \qquad m = \underset{2}{m} = \text{const.}\,, \tag{7}$$

choose a retarded solution of (5), and expand it into an EIH series, then the first non-vanishing radiation terms become

$$\underset{5}{\gamma_{00}} = \frac{4}{3!}\sum \frac{d^3}{dt^3}(mr^2), \qquad \underset{5}{\gamma_{ik}} = 4\sum \frac{d}{dt}(m\dot{\xi}^i\dot{\xi}^k), \tag{8}$$

$$\underset{6}{\gamma_{0k}} = -4\sum m\ddot{\xi}^k - \frac{4}{3!}\sum \frac{d^3}{dt^3}(mr^2\xi^k)\,,$$

where summation is to be taken over all particles. Here, the functions $\xi^k(t)$ are the co-ordinates of a typical particle, $\xi^0 = t$, and r is the Euclidean distance from the particle. The first sum in $\underset{6}{\gamma_{0k}}$ is of at least the 6th order because of the Newtonian equations of motion [2]. The $\underset{5}{g_{00}}$, $\underset{5}{g_{ik}}$ and $\underset{6}{g_{0k}}$ corresponding to (8) can be obtained from the formula

$$\underset{n}{g_{\mu\nu}} = \underset{n}{\gamma_{\mu\nu}} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\varrho\sigma}\underset{n}{\gamma_{\varrho\sigma}}\,.$$

If a *co-ordinate transformation*

$$x^0 = x'^0 + a^0(x'^\nu)\,, \qquad x^k = x'^k + a(x'^\nu)\,,$$

is performed then the first terms affected are

$$\underset{n+2}{g'_{00}} = \underset{n+2}{g_{00}} + 2\underset{n+1}{a_{0,0}}\,, \qquad \underset{n+1}{g'_{0k}} = \underset{n+1}{g_{0k}} + \underset{n+1}{a_{0,k}} + \underset{-n}{a_{k,0}}\,,$$

$$\underset{n}{g'_{ik}} = \underset{n}{g_{ik}} + \underset{n}{a_{i,k}} + \underset{n}{a_{k,i}}\,, \qquad a_\alpha = \eta_{\alpha\beta}a^\beta\,.$$

It can easily be seen that, if $(\underset{n}{g}_{ik}, \underset{n+1}{g}_{0k}, \underset{n+2}{g}_{00})$ is a solution of the field equations, then $(\underset{n}{g}'_{ik}, \underset{n+1}{g}'_{0k}, \underset{n+2}{g}'_{00})$ is also a solution representing the same physical situation in a different co-ordinate system. Because of the Newtonian equations, the functions $\underset{5}{g}_{ik}$ depend on the time only, and thus one can choose $\underset{5}{a}_k$ such that $\underset{5}{g}'_{ik} = 0$. However, the whole set of functions $(\underset{n}{g}_{ik}, \underset{n+1}{g}_{0k}, \underset{n+2}{g}_{00})$ can be annihilated by a co-ordinate transformation only if

(9a) $$\underset{n+2}{g}_{00,ik} + \underset{n}{g}_{ik,00} - \underset{n+1}{g}_{i0,k0} - \underset{n+1}{g}_{k0,i0} = 0\,,$$

(9b) $$\underset{n+1}{g}_{0m,ik} + \underset{n}{g}_{ik,0m} - \underset{n+1}{g}_{0i,km} - \underset{n}{g}_{km,0i} = 0\,,$$

(9c) $$\underset{n}{g}_{im,kl} + \underset{n}{g}_{kl,im} - \underset{n}{g}_{il,km} - \underset{n}{g}_{km,il} = 0\,.$$

That is to say, Eqs. (9) constitute a system of necessary and sufficient conditions for the existence of functions $\underset{n+1}{a}_0$ and $\underset{n}{a}_k$ such that $\underset{n}{g}'_{ik} = \underset{n+1}{g}'_{0k} = \underset{n+2}{g}'_{00} = 0$. Goldberg [5] remarked that by starting with $\underset{n}{g}_{ik} = f_{ik}(t)$ one can obtain solutions of the field equations of the $(n+1)$th and $(n+2)$th orders such that conditions (9) will not be satisfied. However, it must be noted that, since the solutions of the field equations are not unique, one can also start with the same $\underset{n}{g}_{ik}$ and obtain functions $\underset{n+1}{g}_{0k}$ and $\underset{n+2}{g}_{00}$ which can be annihilated. For example, the field

$$\underset{n}{g}_{ik} = f_{ik}(t), \qquad \underset{n+1}{g}_{0k} = \tfrac{1}{2} x^s \dot{f}_{sk}\,, \qquad \underset{n+2}{g}_{00} = 0$$

is flat, i. e. it fulfills conditions (9), but the field

$$\underset{n}{g}_{ik} = f_{ik}(t), \qquad \underset{n+1}{g}_{0k} = 0, \qquad \underset{n+2}{g}_{00} = -r^2 \ddot{f}_{ss}/6$$

is empty and non-flat unless $\ddot{f}_{ik} = \frac{1}{3}\delta_{ik}\ddot{f}_{ss}$ (spherical symmetry), that is

$$\underset{n+2}{g}_{00,ik} + \underset{n}{g}_{ik,00} - \underset{n+1}{g}_{i0,k0} - \underset{n+1}{g}_{k0,i0} = \ddot{f}_{ik} - \tfrac{1}{3}\delta_{ik}\ddot{f}_{ss}\,.$$

The radiation terms $(\underset{3}{g}_{ik} = 0, \underset{4}{g}_{0k} = 0, \underset{5}{g}_{00})$ represent an apparent field, i. e. they can be eliminated by a co-ordinate transformation and do not contribute to the equations of motion up to the 7th order [2]. This suggests that $(\underset{5}{g}_{ik}, \underset{6}{g}_{0k}, \underset{7}{g}_{00})$, should be the first true radiative set contributing to the equations of motion of the 9th order. In the case of a system of pole-particles, $\underset{5}{g}_{ik}$ and $\underset{6}{g}_{0k}$ can be determined from (8), and $\underset{7}{g}_{00}$ from the corresponding field equation

(10) $$\underset{7}{R}_{00} = -\tfrac{1}{2}\Delta \underset{7}{g}_{00} + \tfrac{1}{2}\underset{5}{g}_{00,00} - \tfrac{1}{2}\underset{5}{g}_{ik}\underset{2}{g}_{00,ik} = -8\pi\big(\underset{7}{T}_{00} - \tfrac{1}{2}\underbrace{g_{00}T}_{7}\big)\,,$$

where $\underset{2}{g}_{00}$ is given by the usual EIH expression $\underset{2}{g}_{00} = -\sum 2m/r$. The functions $\underset{7}{T}_{00}$ and $\underset{7}{T}$ can be evaluated from (7). One must note that $m\,dt/ds$ contains a 7th order term $\underset{7}{m} = -\frac{1}{2} m \underset{5}{g}_{00}$. Finally, as the solution of (10), we obtain the expression

(11)
$$\underset{7}{g}_{00} = \frac{2}{5!}\sum \frac{d^5}{dt^5}(mr^4) + \frac{2}{3!}\sum \frac{d^3}{dt^3}(mr^2\dot{\xi}^s\dot{\xi}^s) + \underset{5}{g}_{ik}\sum m r_{,ik} + (\underset{5}{g}_{00} + \tfrac{1}{2}\underset{5}{g}_{ss})\underset{2}{g}_{00}\,.$$

The set of fields $(\underset{5}{g}_{ik}, \underset{6}{g}_{0k}, \underset{7}{g}_{00})$ defined by (8) and (11) cannot, in general, be eliminated by a co-ordinate transformation. This may be confirmed by calculating the expression (9a) for $n = 5$.

4. For spherically symmetric non-rotating particles the equations of motion can be obtained from the "geodesic" equation [7], [8]

(12)
$$m\left(\frac{d^2\xi^\alpha}{d\tilde{s}^2} + \widetilde{\left\{ {\alpha \atop \mu\nu} \right\}}\frac{d\xi^\mu}{d\tilde{s}}\frac{d\xi^\nu}{d\tilde{s}}\right) = 0\,,$$

where $d\tilde{s}^2 = \tilde{g}_{\alpha\beta}d\xi^\alpha d\xi^\beta$, and where $\tilde{\varphi}$ denotes the regular part of φ with x^k replaced by ξ^k (see [8] for details). By eliminating $d\tilde{s}$ from (12) one obtains the equations of motion in the form

(13)
$$\Omega^k \equiv m\left(\ddot{\xi}^k + \left(\widetilde{\left\{ {k \atop \mu\nu} \right\}} - \widetilde{\left\{ {0 \atop \mu\nu} \right\}}\dot{\xi}^k\right)\dot{\xi}^\mu\dot{\xi}^\nu\right) = 0\,.$$

In this notation the Newtonian equations are

$$\underset{4}{\Omega}^k \equiv m\left(\ddot{\xi}^k + \underset{2}{\widetilde{\left\{ {k \atop 00} \right\}}}\right) = 0\,.$$

The equations of motion can be written to the 9th order –

(14)
$$\underset{4}{\Omega}^k + \underset{6}{\Omega}^k + \underset{8}{\Omega}^k + \underset{9}{\Omega}^k = 0\,.$$

Einstein, Infeld and Hoffman [1] have found the first post-Newtonian contribution $\underset{6}{\Omega}^k$. The radiative correction has the form

(15)
$$\underset{9}{\Omega}^k \equiv m\left(\underset{7}{\widetilde{\left\{ {k \atop 00} \right\}}} + 2\underset{6}{\widetilde{\left\{ {k \atop 0s} \right\}}}\dot{\xi}^s + \underset{5}{\widetilde{\left\{ {k \atop rs} \right\}}}\dot{\xi}^r\dot{\xi}^s - \underset{6}{\widetilde{\left\{ {0 \atop 00} \right\}}}\dot{\xi}^k - 2\underset{5}{\widetilde{\left\{ {0 \atop 0s} \right\}}}\dot{\xi}^s\dot{\xi}^k - \underset{4}{\widetilde{\left\{ {0 \atop rs} \right\}}}\dot{\xi}^r\dot{\xi}^s\dot{\xi}^k\right)$$

and can be explicitly evaluated by means of (8) and (11). The explicit form of $\underset{8}{\Omega}^k$ is not known, but it is possible to foresee it in special simple cases.

Let us consider the case of *two bodies* of equal mass m, with coordinates ξ^k and η^k, respectively. The equations of motion up to the 6th order admit a solution which represents the circular motion of these bodies. One may take $\xi^k = -\eta^k$ and

$$\xi^1 = R\cos\omega_0 t, \qquad \xi^2 = R\sin\omega_0 t, \qquad \xi^3 = 0 . \tag{16}$$

The angular velocity ω_0 is a function of R and m which, in the Newtonian approximation, is

$$\omega_0^2 = m/4R^3 . \tag{17}$$

From symmetry considerations it follows that the equations of motion to the 9th order have also a solution with $\xi^k = -\eta^k$. The 8th order terms are either of the form $\underset{8}{A}\xi^k$ or $\underset{7}{B}\dot{\xi}^k$; but $\underset{7}{B} = 0$ by virtue of (16). Thus, the equations to the 8th order also admit periodic solutions representing circular motion. It follows that the equations of motion up to the 9th order have the form

$$m(\ddot{\xi}^k + 2\alpha\dot{\xi}^k + \omega^2\xi^k) = 0 , \tag{18}$$

where α is a constant of the 6th order, and

$$\omega^2 = m/4|\vec{\xi}|^3 + \text{constants of order 4 and 6.} \tag{19}$$

The damping term $2\alpha m\dot{\xi}^k$ arises from $\underset{9}{\Omega}^k$. The value of α is determined by the use of Eq. (15) with $\underset{5}{g_{ik}}, \underset{6}{g_{0k}}, \underset{7}{g_{00}}$ given by (8) and (11). Thus,

$$\alpha = 3m^3/20R^4 . \tag{20}$$

Eq. (18) does not admit exact solutions of the form (16). In spite of the non-linearity of Eq. (18) one might expect it to admit solutions of a "damped" character ($\alpha > 0$), at least for a limited time range.

I am grateful to Professor L. Infeld and Dr. J. Plebański for many interesting discussions during this investigation. The work was concluded at King's College, London, and I am indebted to Professor H. Bondi and Dr. F. A. E. Pirani for their hospitality there.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] A. Einstein, L. Infeld, a. B. Hoffmann, Ann. Math., **39** (1938), 65.
[2] L. Infeld, Phys. Rev., **53** (1938), 836.
[3] N. Hu, Proc. Roy. Irish. Acad., **51** A (1947), 87.
[4] L. Infeld a. A. E. Scheidegger, Can. J. Math., **3** (1951), 195.
[5] J. N. Goldberg, Phys. Rev., **99** (1955), 1873.
[6] L. Infeld, Acta Phys. Polon., **13** (1954), 187.
[7] L. Infeld a. J. Plebański, Bull. Acad. Polon. Sci., Cl. III, **4** (1956), 757.
[8] L. Infeld, Rev. Mod. Phys., **29** (1957), 398.