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GOLDEN OLDIE EDITORIAL

Editorial note to: Władysław Ślebodziński, On Hamilton's canonical equations

Andrzej Trautman

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The article by W. Ślebodziński, included here in English translation, has become a classic in tensor calculus mainly because it contains, for the first time in print, the general formula for—what is now called—the Lie derivative[1] $\mathcal{L}(X)T$ of a tensor field T with respect to a vector field X. In special cases, the formula was known and used before. In particular, D. Hilbert had in [1] an expression equivalent to the Lie derivative of a metric tensor g. Around 1920, É. Cartan defined the Lie derivative of exterior differential forms; see equation (5) on p. 84 in [2]. In connection with this definition, Ślebodziński gives a reference to a paper by Th. Lepage published in 1929. D. van Dantzig recognized the priority of Ślebodziński in defining the Lie derivative in the general case, introduced the name *Liesche Ableitung* and complemented the work of Ślebodziński by pointing out that the Lie derivative with respect to a vector field X of a geometric object T can be defined as the difference between the value of T at a point and the value of that object at the same point obtained by an infinitesimal 'dragging along' the vector field X [3]. In contemporary notation this is expressed by

$$\mathcal{L}(X)T = \frac{d}{dt}\varphi_t^* T_{|t=0},$$

[1] In this note, I transcribe equations from the form given by the Author to the notation in current usage.

The republication of the original paper can be found in this issue following the editorial note and online via doi:10.1007/s10714-010-1057-6.

A. Trautman (✉)
Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, Warsaw, Poland
e-mail: andrzej.trautman@fuw.edu.pl

where $\varphi_t^* T$ is the pull-back of T by the flow $(\varphi_t, t \in \mathbb{R})$ generated by X. In view of the equation

$$\frac{d}{dt}\varphi_t^* T = \varphi_t^* \mathcal{L}(X) T,$$

the vanishing of $\mathcal{L}(X)T$ is equivalent to the invariance of T with respect to the flow generated by X: local invariance is reduced to the infinitesimal one; see, e.g., §24 in [4] or §4.3 in [5]. In his influential book J. A. Schouten, in footnote 1 on p. 102 of [6], lists the 1931 paper by Ślebodziński as the first reference for the notion of Lie differentiation.

In the first section of the article, the most important is equation (4) defining the Lie derivative with respect to X of a mixed tensor. The author notes that the Lie derivative satisfies the Leibniz rule for tensor products and that it commutes with contractions over pairs of contragredient tensor indices.

Section 2 introduces the main theme of the article: Hamilton's canonical equations and the integral invariants in the sense of Poincaré [7]. The Hamiltonian is considered there as a function H on the (phase) space $\mathbb{R}^{2n}$ with coordinates $(x^i)_{i=1,\dots,2n}$ and the (symplectic) 2-form (10), written here as

$$\omega = \frac{1}{2} A_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j, \tag{*}$$

where $A_{ij} + A_{ji} = 0$, $A_{i\,n+i} = -A_{n+i\,i} = 1$ for $i = 1, \dots, n$ and, if $|i - j| \neq n$, then $A_{ij} = 0$. Physicists usually write $(q^1, \dots, q^n) = (x^1, \dots, x^n)$ and $(p_1, \dots, p_n) = (x^{n+1}, \dots, x^{2n})$ so that $\omega = \mathrm{d}q^\mu \wedge \mathrm{d}p_\mu$ (sum over $\mu = 1, \dots, n$). The Hamiltonian vector field X_H, given by the Author in eq. (9'), is nowadays defined by its contraction with ω

$$X_H \lrcorner\, \omega = \mathrm{d}H, \quad \text{thus, in physicists' notation,} \quad X_H = \frac{\partial H}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial H}{\partial q^\mu}\frac{\partial}{\partial p_\mu}$$

The Lie derivatives with respect to X_H of ω, of the tensor A defined in (*) and of its inverse B all vanish. The flow of X_H solves the canonical equations of motion.

Section 3, entitled *Generalizations*, is devoted to a few simple results in the theory of integral invariants. The invariants are given by tensors with vanishing Lie derivatives with respect to X_H. Using the tensors A and B to raise, lower and contract indices of such tensors, and by forming their products, one obtains new tensors with the same property of invariance. This procedure generalizes the results described by de Donder [8].

Further information on the history of the notion of Lie differentiation and its generalizations, and on their applications in mathematical physics, can be found in [9–12] and in the bibliographies given there.

Władysław Ślebodziński—a brief biography

By Witold Roter

Władysław Ślebodziński was born on 6th February 1884 in Pysznica, district Nisko. He studied mathematics at the Jagiellonian University in Kraków in the years 1903–1908. After graduation he was a teacher of mathematics for the next 5 years. Owing to Kretkowski's foundation, he spent the academic year 1913/14 in Göttingen. The outbreak of the first world war forced him to return home. At the beginning of 1919 he moved from Kraków to Poznań, where he began to lecture in the State Higher School of Machine Construction and Electric Technology. In Poznań he prepared the doctor and habilitation dissertations, defended at Warsaw University in 1929 and 1934, respectively.

During the German occupation he was expelled from Poznań and began clandestine teaching. Denunciated, he was arrested in 1942. In January 1943 he was transported to the concentration camp Auschwitz, from where he was sent to Gross-Rosen and afterwards to the then-newly established concentration camp in Nordhausen (Thüringen). There he lived to see freedom. In July 1945 he returned to Poland. Settling in Wrocław, he took up the post of a professor at the, at that time, joint Wrocław's schools of higher education: the University and the University of Technology. Here he was one of the organizers of the new Wrocław mathematical center.

Professor W. Ślebodziński was an excellent mathematician. His scientific interests included the theory of Riemannian spaces, the theory of surfaces in affine spaces, exterior forms, problems of equivalence, integral invariants, infinite Lie groups and manifolds of almost complex structure. He wrote about 50 scientific papers, two monographs on exterior forms and their applications (a 2-volume monograph in French, and its revised and expanded version in English), as well as many review articles. His results always dealt with the main trends in differential geometry.

One of his greatest achievements was the introduction (in the paper [13], reprinted in this issue) of a new differential operator which can be applied to scalars, tensors and affine connections. Later (1932) the operator was called the Lie derivative by van Dantzig [3].

The outstanding Japanese geometer Kentaro Yano devoted to the notion a monograph [9], where one can find the definition of the Lie derivative (of general geometric objects), its basic properties and many applications. Apparently Yano's book contains all the results published on the subject up to 1955. He studied there groups of motions, groups of affine motions, groups of projective and conformal motions. Moreover, he considered also the Lie derivatives in general affine spaces of geodesics, in compact orientable Riemannian manifolds, and almost complex manifolds.

Thus, the Lie derivative contributed to quick development of many areas of differential geometry. It is an important and powerful instrument in the study of groups of automorphisms and has extensive and important application in geometry. It belongs to very useful and important notions in differential geometry.

It seems that D. van Dantzig was the first who applied the notion of the Lie derivative to physics. More precisely, he applied the Lie derivative to thermo-hydrodynamics of ideal fluids [14]. The Lie derivatives play now an important role in mathematical physics in connection with conservation laws and the study of symmetries of solutions of field equations such as Einstein's equations of general relativity.

For his contributions Prof. Ślebodziński was many times rewarded. He received honorary doctorates from the Wrocław University and the Technical Universities of Wrocław and Poznań.

He died on the 3rd January 1972 in Wrocław.

Professor Władysław Ślebodziński's contributions have recently received a new recognition. In February 2009 the city of Wrocław honoured Ślebodziński by naming one of the new streets after him.

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Witold Roter, Professor Emeritus
Institute of Mathematics and Computer Science,
Wrocław University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław,
Poland

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