

Pin^c and Lipschitz structures on products of manifolds

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Abstract. The topological condition for the existence of a pin^c structure on the product of two Riemannian manifolds is derived and applied to construct examples of manifolds having the weaker Lipschitz structure, but no pin^c structure. An example of a five-dimensional manifold with this property is given; it is pointed out that there are no manifolds of lower dimension with this property.

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1. Introduction

There are two main approaches to complex spinors on Riemannian manifolds that are not necessarily orientable. The first assumes the existence of a pin^c structure, defined as a reduction of the bundle of all orthonormal frames of a Riemannian manifold M of dimension m to the group $\text{Pin}^c(m)$. The consideration of a pin^c , rather than pin , structure is motivated, in part, by the need of physics to describe *charged* fermions. The insistence on pin , rather than spin , structures is also influenced by physics where it is necessary to consider transformations of spinor fields under reflections. Spin^c structures appear in the theory of the Seiberg–Witten invariants. The second approach is based on the notion of complex *spinor bundles*. If m is even, then these two approaches are equivalent: the spinor bundle is associated with a pin^c structure. If m is odd and M is not orientable, then a spinor bundle is associated with a weaker *Lipschitz structure*, a notion introduced by Friedrich and Trautman [5]; it is recalled here in Section 2.

In every mathematical category it is of interest to know for which pairs of objects a product is defined. For example, there is a natural,

direct product of every two Riemannian manifolds. If both factors have a spin structure ('are spin'), then their product is also spin. The product of two Riemannian manifolds has a pin structure if, and only if, both factors are pin and at least one of them is orientable [2].

In this paper we consider the question of existence of a pin^c structure on the product of two closed (i.e. compact without boundary) Riemannian manifolds. The main result consists in the proof, in Section 3, of the following

Theorem. *Let M_1 and M_2 be closed Riemannian manifolds. The Riemannian manifold $M_1 \times M_2$ has a pin^c structure if, and only if, both M_1 and M_2 have a pin^c structure and there holds one of the following conditions:*

- (i) *one of the manifolds is orientable,*
- (ii) *the first Stiefel–Whitney classes of both manifolds have integral lifts, i.e. there exist elements $c_i \in H(M_i, \mathbb{Z})$, $i = 1, 2$, such that*

$$w_1(TM_i) \equiv c_i \pmod{2} \quad \text{for } i = 1, 2.$$

To make the paper self-contained, in Section 2 we recall some results on Lipschitz structures proved in [5]. In Section 4, the Theorem is used to construct examples of manifolds with a Lipschitz structure that do not admit a pin^c structure. Among them is a manifold of dimension 5, this being the lowest dimension for which this phenomenon can occur.

2. Preliminaries

2.1. NOTATION AND TERMINOLOGY

We use the standard notation and terminology, mainly the one of [4], [5] and [8]. For the convenience of the reader, some of the relevant definitions are presented below. The ring of integers modulo m , where $1 < m \in \mathbb{N}$, is denoted by \mathbb{Z}/m ; in particular, $\mathbb{Z}/2$ is the two-element field. A real, finite-dimensional vector space V with a positive-definite quadratic form is referred to as a *Euclidean space*; the orthogonal group of its automorphisms is $O(V)$. The dual of a finite-dimensional vector space S over \mathbb{C} is $S^* = \text{Hom}(S, \mathbb{C})$ and $S \times S^* \ni (\varphi, \omega) \mapsto \langle \varphi, \omega \rangle \in \mathbb{C}$ is the evaluation map. The vector space of all semi-linear maps from S^* to \mathbb{C} is denoted by \bar{S} ; there is a semi-linear map $S \rightarrow \bar{S}$, $\varphi \mapsto \bar{\varphi}$, defined by $\langle \omega, \bar{\varphi} \rangle = \overline{\langle \varphi, \omega \rangle}$ for every $\omega \in S^*$. The vector spaces $\text{Hom}(\bar{S}, \mathbb{C})$ and $\overline{\text{Hom}(S, \mathbb{C})}$ can be identified and are denoted by \bar{S}^* . All manifolds and maps among them are assumed to be smooth; TM denotes the total space of the tangent bundle of the manifold M . The trivial bundle $M \times \mathbb{R}^k$ is denoted by θ^k .

2.2. RESULTS FROM TOPOLOGY

Our main references for algebraic topology are [6] and [9]. The standard notation for the homology and cohomology groups is used; the i th Stiefel–Whitney class of a vector bundle $E \rightarrow M$ is denoted by $w_i(E)$.

2.2.1. Obstructions to existence of spinor structures

Recall that $w_1(TM) = 0$ is equivalent to the orientability of M . Depending on whether the group Pin is defined in terms of a Clifford algebra based on a quadratic form that is positive- or negative-definite, one has the notion of a pin^+ or pin^- structure. A Riemannian manifold M has a pin^+ (resp., pin^-) structure if, and only if, $w_2(TM) = 0$ (resp., $w_2(TM) + w_1(TM)^2 = 0$); it has a pin^c structure if, and only if, $w_2(TM)$ is the mod 2 reduction of an integral cohomology class [4, 8].

2.2.2. The Universal Coefficient Theorems

These theorems relate homology and cohomology groups with different coefficients. Let R be a ring; there are natural short exact sequences:

$$0 \rightarrow H_n(X, \mathbb{Z}) \otimes R \rightarrow H_n(X, R) \rightarrow \text{Tor}(H_{n-1}(X, \mathbb{Z}), R) \rightarrow 0$$

and

$$0 \rightarrow \text{Ext}(H_{n-1}(X, \mathbb{Z}), R) \rightarrow H^n(X, R) \rightarrow \text{Hom}(H_n(X, \mathbb{Z}), R) \rightarrow 0.$$

When R is a field, then

$$H^n(X, R) \cong \text{Hom}(H_n(X, R), R).$$

2.3. PIN^c AND LIPSCHITZ GROUPS

2.3.1. Clifford algebras and spinor representations

Let V be a Euclidean, m -dimensional space, and let $\text{Cl}(V)$ be its Clifford algebra. We put $\text{Cl}^c(V) = \mathbb{C} \otimes \text{Cl}(V)$. A volume element $\eta \in \text{Cl}^c(V)$, associated with V , is defined as the product of a sequence of m pairwise orthogonal vectors and normalized so that $\eta^2 = 1$; if η is a volume element, then so is $-\eta$. There is a canonical antiautomorphism $a \mapsto a^\dagger$ of $\text{Cl}^c(V)$ defined as an \mathbb{R} -linear isomorphism of the vector structure such that $a^\dagger = \bar{a}$ for $a \in \mathbb{C} \subset \text{Cl}^c(V)$, $v^\dagger = v$ for $v \in V \subset \text{Cl}^c(V)$ and $(ab)^\dagger = b^\dagger a^\dagger$ for every a and $b \in \text{Cl}^c(V)$. The antiautomorphism defines a unitary group

$$\text{U}(V) = \{a \in \text{Cl}^c(V) \mid a^\dagger a = 1\}.$$

We assume from now on that V is of odd dimension $2n - 1$, $0 < n \in \mathbb{N}$, define W to be the *orthogonal* sum $V \oplus \mathbb{R}$ and denote by $u \in W$ a

unit vector orthogonal to V . Since the dimension $2n$ of W is even, the algebra $\text{Cl}^c(W)$ is simple and there is a complex, 2^n -dimensional vector space S and an isomorphism

$$(1) \quad \gamma : \text{Cl}^c(W) \rightarrow \text{End}S$$

of complex algebras with units, i.e., a faithful and irreducible ‘Dirac representation’ of the Clifford algebra in S . The map $a \mapsto \overline{\gamma(a^\dagger)^*} \in \text{End}\bar{S}^*$ is also a faithful irreducible representation of $\text{Cl}^c(W)$; the simplicity of the algebra implies that they are equivalent: there is an isomorphism $\Phi : \bar{S} \rightarrow S^*$ such that $\overline{\gamma(a^\dagger)} = \Phi^{-1}\gamma(a)^*\Phi$. By rescaling, Φ can be made to satisfy $\Phi^* = \bar{\Phi}$; the Hermitean form

$$(2) \quad (\varphi|\psi) = \langle \varphi, \bar{\Phi}\psi \rangle, \quad \varphi, \psi \in S,$$

is positive and invariant with respect to the action of the group $U(W)$ in S , $(\gamma(a)\varphi|\gamma(a)\varphi) = (\varphi|\varphi)$ for every $a \in U(W)$ and $\varphi \in S$. The scalar product (2) is used to define the adjoint f^\dagger of $f \in \text{End}S$ so that $\gamma(a^\dagger) = \gamma(a)^\dagger$ for every $a \in \text{Cl}^c(W)$.

The restriction of (1) to $\text{Cl}^c(V) \subset \text{Cl}^c(W)$ is a faithful, but reducible, representation of $\text{Cl}^c(V)$ in S : if η is the volume element associated with V , then the subspaces

$$(3) \quad S_\pm = \{\varphi \in S \mid \gamma(\eta)\varphi = \pm\varphi\}$$

are invariant for this restriction and $\gamma|_{\text{Cl}^c(V)} = \sigma_+ \oplus \sigma_-$. The ‘Pauli representations’ σ_+ and σ_- of $\text{Cl}^c(V)$ in S_+ and S_- , respectively, are irreducible and inequivalent, but not faithful. For the purposes of this paper, it is convenient to refer to γ and $\gamma|_{\text{Cl}^c(V)}$ as the *spinor representations* of the algebras in question.

The notation is chosen so that, taking into account that γ is an isomorphism, one can *identify* $\text{Cl}^c(W)$ with $\text{End}S$ and this is done in the sequel.

2.3.2. *Pin^c groups*

The group $\text{Pin}(W) \subset \text{Cl}(W)$ is generated by the set of all unit elements of W and $\text{Pin}^c(W) \subset \text{Cl}^c(W)$ is generated by $\text{Pin}(W)$ and U_1 . The adjoint representation of $\text{Pin}^c(W)$ in W is given by $\text{Ad}(a)v = av a^{-1}$. Similar definitions apply to the groups $\text{Pin}(V)$ and $\text{Pin}^c(V)$.

The dimension of W being even, there is the exact sequence

$$(4) \quad 1 \rightarrow U_1 \rightarrow \text{Pin}^c(W) \xrightarrow{\text{Ad}} O(W) \rightarrow 1.$$

The volume elements of the odd-dimensional space V are in the kernel of Ad and the image of $\text{Pin}^c(V)$ by Ad is the special orthogonal group

$\mathrm{SO}(V)$. To describe spinors on a non-orientable manifold, one needs a sequence like (4), with the full orthogonal group as the image of Ad . In odd dimensions, this can be achieved by either using, instead of Ad , the *twisted adjoint representation* [1] or extending the group $\mathrm{Pin}^c(V)$ to a larger *Lipschitz group*. Only the latter approach is suitable when one considers spinor bundles as the primitive notion.

2.3.3. The Lipschitz group

In the notation of §2.3.1, the Lipschitz group, associated with the odd-dimensional Euclidean space $V \subset W$, is

$$\mathrm{Lpin}(V) = \{a \in \mathrm{GL}(S) \mid a^\dagger a = 1 \quad \text{and} \quad aVa^{-1} = V\}.$$

Clearly, $\mathrm{Pin}^c(V)$ is a subgroup of $\mathrm{Lpin}(V)$ and the homomorphism $\mathrm{Ad} : \mathrm{Lpin}(V) \rightarrow \mathrm{O}(V)$ is surjective because $u \in \mathrm{Lpin}(V)$ and $\mathrm{Ad}(u) = -\mathrm{id}_V$. The kernel of Ad is the subgroup

$$\left\{ \frac{1}{2}(1 + \eta)z_+ + \frac{1}{2}(1 - \eta)z_- \mid z_+, z_- \in \mathrm{U}_1 \right\} \cong \mathrm{U}_1 \times \mathrm{U}_1$$

so that there is an exact sequence

$$1 \rightarrow \mathrm{U}_1 \times \mathrm{U}_1 \rightarrow \mathrm{Lpin}(V) \xrightarrow{\mathrm{Ad}} \mathrm{O}(V) \rightarrow 1.$$

2.4. SPINOR BUNDLES AND LIPSCHITZ STRUCTURES

2.4.1. Spinor structures

Let M be a Riemannian manifold of dimension m ; let V be a Euclidean space of dimension m and denote by P the $\mathrm{O}(V)$ -bundle of orthonormal frames on M . Let us agree to say that G is a *spinor group* if it contains $\mathrm{Spin}(V)$ as a subgroup and there is homomorphism $\rho : G \rightarrow \mathrm{O}(V)$ such that $\rho(\mathrm{Spin}(V)) = \mathrm{SO}(V)$. A *spinor structure* of type (G, ρ) on M is a reduction Q of P to a spinor group G characterized by the maps

$$(5) \quad \begin{array}{ccc} G & \longrightarrow & Q \\ \rho \downarrow & & \downarrow \chi \\ \mathrm{O}(V) & \longrightarrow & P \longrightarrow M \end{array}$$

such that $\chi(qa) = \chi(q)\rho(a)$ for $q \in Q$ and $a \in G$.

2.4.2. Spinor bundles

One defines total space of the complex Clifford bundle associated with a Riemannian manifold M as

$$\mathrm{Cl}^c(TM) = \bigcup_{x \in M} \mathrm{Cl}^c(T_x M).$$

A complex vector bundle $\Sigma \rightarrow M$, with a homomorphism

$$\tau : \text{Cl}^c(TM) \rightarrow \text{End} \Sigma$$

of bundles of algebras over M , is said to be a bundle of Clifford modules. In particular, a *spinor bundle* is a bundle of Clifford modules such that the restriction of τ to every fibre is a spinor representation in the sense of §2.3.1.

A spinor bundle Σ on an odd-dimensional manifold is said to be *decomposable* if there is a non-trivial vector bundle decomposition $\Sigma = \Sigma_+ \oplus \Sigma_-$ such that $\tau(\text{Cl}^c(TM))\Sigma_{\pm} \subset \Sigma_{\pm}$. In [5] it is shown that there holds

Proposition 1. *A spinor bundle on an odd-dimensional Riemannian manifold M is decomposable if, and only if, M is orientable.*

Note that, for every $x \in M$, there is a decomposition $\Sigma_x = \Sigma_{x+} \oplus \Sigma_{x-}$, as in (3), and $\tau(\text{Cl}^c(T_x M))\Sigma_{x\pm} \subset \Sigma_{x\pm}$, but a *global* decomposition holds only in the orientable case.

Since, by assumption, M is paracompact (even compact), there is a Hermitian scalar product on the fibres of $\Sigma \rightarrow M$,

$$(\cdot | \cdot)_{\Sigma} : \Sigma \times_M \Sigma \rightarrow \mathbb{C}.$$

2.4.3. Lipschitz structures

Let M and V be a Riemannian manifold and a Euclidean space, both of odd dimension m , respectively. A *Lipschitz structure* on M is defined as a spinor structure (5) of type $(\text{Lpin}(V), \text{Ad})$. A spinor bundle $\Sigma \rightarrow M$ with a Hermitian scalar product defines, and is associated with, a Lipschitz structure such that the fibre Q_x , $x \in M$, consists of all isomorphisms $q : S \rightarrow \Sigma_x$ satisfying $qVq^{-1} = \tau(T_x M)$ and $(q\varphi|q\varphi)_{\Sigma} = (\varphi|\varphi)$ for every $\varphi \in S$. The action of $\text{Lpin}(V)$ on Q is by composition of maps.

Proposition 2. *An odd-dimensional manifold M admits a Lipschitz structure if, and only if, there is a vector bundle $\mathbb{R}^2 \rightarrow E \rightarrow M$ and $c \in H^2(M, \mathbb{Z})$ such that*

$$(6) \quad w_2(TM) + w_2(E) \equiv c \pmod{2}.$$

A proof of this proposition is in [5].

3. Proof of the Theorem

Throughout the rest of the paper, M_1 and M_2 are closed Riemannian manifolds. By virtue of the Künneth theorem, every element $\alpha \in H^2(M_1 \times M_2, \mathbb{Z}/2)$ admits a decomposition

$$(7) \quad \alpha = \alpha_{20} + \alpha_{11} + \alpha_{02}, \quad \alpha_{ij} \in H^i(M_1, \mathbb{Z}/2) \otimes H^j(M_2, \mathbb{Z}/2).$$

Lemma. *An element $\alpha \in H^2(M_1 \times M_2, \mathbb{Z}/2)$, decomposed as in (7), is the mod 2 reduction of $c \in H^2(M_1 \times M_2, \mathbb{Z})$ if, and only if, there are elements $c_{ij} \in H^i(M_1, \mathbb{Z}) \otimes H^j(M_2, \mathbb{Z})$ such that $\alpha_{ij} \equiv c_{ij} \pmod{2}$.*

Proof. From the second sequence in §2.2.2 and the fact that $H_0(M, \mathbb{Z})$ is free there follows

$$(8) \quad H^1(M, R) \cong \text{Hom}(H_1(M, \mathbb{Z}), R).$$

Therefore, the groups $H^i(M, \mathbb{Z})$ are free for $i = 0$ and 1; the Künneth theorem applied to c gives

$$c = c_{20} + c_{11} + c_{02}, \quad c_{ij} \in H^i(M_1, \mathbb{Z}) \otimes H^j(M_2, \mathbb{Z}).$$

To complete the proof, one reduces both sides of the last equality modulo 2 and notes that such a reduction respects the decomposition and acts on each summand separately. \square

The Whitney formula gives

$$w_2(T(M_1 \times M_2)) = w_2(TM_1) \otimes 1 + w_1(TM_1) \otimes w_1(TM_2) + 1 \otimes w_2(TM_2).$$

Applying the Lemma to the last equation one obtains that a *necessary* condition for the existence of a pin^c structure on $M_1 \times M_2$ is that both M_1 and M_2 have pin^c structures. The condition becomes *sufficient* if, in addition, there holds:

$$(9) \quad \alpha_{11} = w_1(TM_1) \otimes w_1(TM_2) \quad \text{is the mod 2 reduction} \\ \text{of an element } c_{11} \in H^1(M_1, \mathbb{Z}) \otimes H^1(M_2, \mathbb{Z}).$$

To complete the *proof of the Theorem*, one has to show that the alternative ‘(i) or (ii)’ is equivalent to (9). Part (i) is easy: one of the manifolds under consideration is orientable, if and only if, one can take $c_{11} = 0$ in (9). Assume now that (i) does not hold so that $\alpha_{11} \neq 0$. Since our manifolds are closed, their homology and cohomology groups are finitely generated,

$$(10) \quad H_1(M_1, \mathbb{Z}) \cong \mathbb{Z}^k \oplus \mathbb{Z}/2^{p_1} \oplus \dots \oplus \mathbb{Z}/2^{p_s} \oplus T_1,$$

$$(11) \quad H_1(M_2, \mathbb{Z}) \cong \mathbb{Z}^l \oplus \mathbb{Z}/2^{q_1} \oplus \dots \oplus \mathbb{Z}/2^{q_t} \oplus T_2,$$

where T_i is the torsion of $H_1(M_i, \mathbb{Z})$ other than the 2-torsion part so that $T_i \otimes_{\mathbb{Z}} \mathbb{Z}/2 = 0$. Using (8) one can write

$$\begin{aligned} H^1(M_1, \mathbb{Z}) &= \mathbb{Z}^k && \text{with basis } (a_1, \dots, a_k), \\ H^1(M_2, \mathbb{Z}) &= \mathbb{Z}^l && \text{with basis } (b_1, \dots, b_l), \\ H^1(M_1, \mathbb{Z}/2) &= \mathbb{Z}/2^k \oplus \mathbb{Z}/2^s && \text{with basis } (a'_1, \dots, a'_k, c'_1, \dots, c'_s), \\ H^1(M_2, \mathbb{Z}/2) &= \mathbb{Z}/2^l \oplus \mathbb{Z}/2^t && \text{with basis } (b'_1, \dots, b'_l, d'_1, \dots, d'_t), \end{aligned}$$

where a'_i and b'_i is the reduction mod 2 of a_i and b_i , respectively. The classes $w_1(TM_1)$ and $w_1(TM_2)$ can be decomposed with respect to the above bases and one sees that condition (9) is satisfied if, and only if, $w_1(TM_1)$ and $w_1(TM_2)$ do not contain summands with c'_i and d'_i , respectively. \square

4. Examples

4.1. PRODUCTS OF PIN^c MANIFOLDS

Example 1. Recall that the real projective space \mathbb{RP}_{2k} , $k \in \mathbb{N}$, has a pin structure [3]; therefore, *a fortiori*, it has a pin^c structure. If $M_1 = \mathbb{RP}_{2k}$ and $M_2 = \mathbb{RP}_{2l}$, where k and l are positive integers, then $M_1 \times M_2$ has no pin^c structure.

Proof. Indeed, $w_1(TM_1) \otimes w_1(TM_2) \neq 0$ cannot be the reduction of an integral cohomology element because $H^1(\mathbb{RP}_n, \mathbb{Z}) = 0$ for $n \geq 2$. \square

Example 2. For every integer $k \geq 5$ we now construct a non-orientable manifold M_k of dimension k so that $M_k \times M_k$ has a pin^c structure. Let B be the 2-dimensional Möbius band and let D_k be the unit ball of dimension k so that its boundary is \mathbb{S}_{k-1} . The manifold M_k is defined as the sum of two manifolds, glued along their common boundary,

$$M_k = (B \times \mathbb{S}_{k-2}) \cup_{\mathbb{S}_1 \times \mathbb{S}_{k-2}} (\mathbb{S}_1 \times D_{k-1}).$$

The non-orientability of B implies that of M_k . One easily finds

$$H_0(M_k, \mathbb{Z}) = \mathbb{Z}, \quad H_1(M_k, \mathbb{Z}) = \mathbb{Z}, \quad H_2(M_k, \mathbb{Z}) = 0 \quad \text{for } k \geq 5.$$

These groups are free and so are the groups $H_i(M_k \times M_k, \mathbb{Z})$ for $i = 0, 1, 2$. This implies that all elements of $H^2(M_k \times M_k, \mathbb{Z}/2)$ have an integral lift; therefore, the manifold $M_k \times M_k$ has a pin^c structure, but, being a product of two non-orientable manifolds, it has no pin structure.

4.2. LIPSCHITZ MANIFOLDS WITHOUT pin^c STRUCTURE

In this section, we construct a series of examples of manifolds with a Lipschitz structure and without pin^c structure. Following a remark in [7], we show that every three-dimensional manifold is pin^- . We construct examples of five-dimensional manifolds with a Lipschitz structure and without pin^c structure.

Example 3. From Example 1 it follows that $M = \mathbb{R}P_2 \times \mathbb{R}P_2$ has no pin^c structure. Therefore, the same is true of the five-dimensional manifold $N = M \times \mathbb{S}_1$. We now construct a vector bundle $E \rightarrow N$, with fibres of real dimension 2, such that $TN \oplus E$ is a trivial. The embedding $\mathbb{R}P_2 \rightarrow \mathbb{R}P_3 \cong \text{SO}_3$ induces on $\mathbb{R}P_2$ a trivial bundle so that there is a line bundle l such that $T\mathbb{R}P_2 \oplus l$ is trivial. One puts $E = l \oplus l$. From the Whitney product theorem one obtains $w_1(E) = w_1(TN)$ and

$$w_2(TN) + w_2(E) = w_1(TN)^2.$$

Since, for every manifold M , $w_1(TM)^2$ is the reduction mod 2 of an integral element (see, e.g., Section 6 in [5]), the criterion (6) for the existence of a Lipschitz structure on N is satisfied.

If M is a *three*-dimensional manifold, then Wu's formula (see §11 in [9]) gives

$$(12) \quad w_1(TM) = \text{Sq}^0(v_1) + \text{Sq}^1(v_0) = v_1$$

$$(13) \quad w_2(TM) = \text{Sq}^0(v_2) + \text{Sq}^1(v_1) + \text{Sq}^2(v_0) = v_2 + (v_1)^2$$

$$(14) \quad v_2 = 0.$$

Here $v_i \in H^i(M, \mathbb{Z}/2)$ is the i th Wu class of M characterized by

$$v_i \cup w = \text{Sq}^i(w) \text{ for every } w \in H^{3-i}(M, \mathbb{Z}/2)$$

so that $v_i = 0$ for $i > 3 - i$ implying (14). Equations (12)-(14) give now $w_2(TM) = w_1(TM)^2$; this shows that every three-dimensional manifold has pin^- structure; therefore, *a fortiori*, a pin^c structure (§2.2.1). As a consequence, five is the smallest dimension of a manifold admitting a Lipschitz structure, but no pin^c structure.

Example 3 can be generalized; it suffices to note that the example is based on two facts: $\mathbb{R}P_3$ is parallelizable and $T\mathbb{S}_1$ is stably trivial (even trivial). The projective space $\mathbb{R}P_7$ is also parallelizable and there are many manifolds with a stably trivial tangent bundle (Lie groups, spheres, orientable three-dimensional manifolds and products of these manifolds). This leads to

Example 4. If M is $(2n + 1)$ -dimensional and such that

$$TM \oplus \theta^{2(k+l+1)} \cong \theta^{2(k+l+n+1)+1} \quad \text{where } k, l \in \{1, 3\},$$

then the manifold

$$(15) \quad \mathbb{R}P_{2k} \times \mathbb{R}P_{2l} \times M$$

has a Lipschitz structure, but no pin^c structure. Indeed, denoting by l_i the normal line bundle for the embedding $\mathbb{R}P_{2i} \rightarrow \mathbb{R}P_{2i+1}$, one can take

$$E = l_k \oplus l_l.$$

One easily checks that

$$T(\mathbb{R}P_{2k} \times \mathbb{R}P_{2l} \times M) \oplus l_k \oplus l_l \cong TM \oplus \theta^{2(k+l+1)} \cong \theta^{2(k+l+m+1)+1}.$$

and so the manifold (15) has the desired property.

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