Spinors in geometry and physics *

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I have known Paolo Budinich for over 30 years; in the 1980s, thanks to the hospitality extended to me by SISSA in Trieste, I had the opportunity to collaborate with him on spinors. I was much impressed by his love and knowledge of the subject. For me, he demonstrated a good taste in science by his fascination with the work of Élie Cartan. It was Paolo who drew my attention to Cartan's simple—nowadays called *pure*—spinors. We wrote together a few papers [5–7] and *The Spinorial Chessboard* [E], a booklet on Clifford algebras and their representations. I very much admire Paolo Budinich for what he did to make Trieste into a scientific centre of international reputation. We also became personal friends, spending together much time on the Carso and in places such as Cortina d'Ampezzo and Zielonowo, a tiny village in Poland.

With great pleasure I dedicate this review to Paolo Budinich, my friend of many years, always so young in spirit.

1. INTRODUCTION

Spinors permeate all of modern physics and have an important place in mathematics. Several books and reviews have been written on the subject; some of them are listed at the end of the article. I present here only a brief, personal view of this field; perhaps it may be of some use as a guide to literature for students and young researchers. Spinors involve subtle mathematics; I try here to be as little technical as possible.

Only some aspects of spinors are outlined here. Spinors in physics are a very vast and rather well-known subject; for this reason, in Section 3 only a few key words are given. Much space is devoted to the algebra of spinors and null elements, reflecting Paolo Budinich's interest in pure spinors: this notion is reviewed in Section 5. In Section 7, I review some applications of spinor notions in general relativity theory: this corresponds to my own interests. While writing this article, I freely used material from my earlier publications listed in the bibliography. Many important aspects of spinors are left aside. In particular, spinor fields are treated in the physicists', local manner, instead of being defined in terms of spin structures [A,I,J]. Moreover, topological aspects and results, such as the Atiyah–Singer index theorem for the Dirac operator $\mathcal{D}$ and vanishing theorems for

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$\mathcal{D}$ [J] are completely left out. Similar remarks apply to estimates of the eigenvalues of $\mathcal{D}$ and the Seiberg–Witten invariants [I], [15]. Nothing is said here about twistors [K], Killing spinors [B], triality and octonions [F,G], [1,12,30]. Another major topic that is neglected here is that of spinors in supersymmetric and string theories [L]. The books [I] and [J] contain many references to recent literature.

The *notation* used in this article is standard in differential geometry and mathematical physics. The exterior algebra associated with a vector space W is $\wedge W$; the symbols $\otimes$, $\wedge$ and $\lrcorner$ denote the tensor, exterior and interior products, respectively. I write also $e(v)w = v \wedge w$ and $i(v)w = v \lrcorner w$ for $v \in W$ and $w \in \wedge W$. The map $(w, w') \mapsto \langle w, w' \rangle$ is the evaluation map of the 1-form w' on the vector w. If $f \in \operatorname{End} W$, then $f^* \in \operatorname{End} W^*$ is defined by $\langle w, f^*(w') \rangle = \langle f(w), w' \rangle$ for every $w \in W$ and $w' \in W^*$. I use the Einstein summation convention over repeated indices. If $N \subset V$, then $N^\perp$ is the set of all elements of V orthogonal to every element of N.

2. HISTORICAL REMARKS

There is a prehistory of spinors: they appear, in disguise, in Euclid's solution

$$x = p^2 - q^2, \quad y = p^2 + q^2, \quad z = 2pq, \quad p, q \in \mathbb{N}, \tag{1}$$

of the Pythagorean equation $x^2 + z^2 = y^2$. The solution (1) is equivalent to

$$\begin{pmatrix} y + x & z \\ z & y - x \end{pmatrix} = 2 \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} p, & q \end{pmatrix},$$

an equation that represents the 'null vector' (x, y, z) as the square of the 'spinor' (p, q); see [33] for further remarks. In Euler's formulae for the rational representation of rotations in $\mathbb{R}^3$ one can find—with a little effort and some good will—the map $\mathrm{SU}_2 \to \mathrm{SO}_3$. Hamilton represented rotations in terms of quaternions: every rotation in $\mathbb{R}^3$ is of the form $q \mapsto aqa^{-1}$, where $a \in \mathrm{Sp}_1$ and $q \in \mathbb{R}^3 \subset \mathbb{H}$ are a unit and a pure quaternion, respectively (this shows that the groups Spin_3 and Sp_1 are isomorphic). Cayley (1855) extended Hamilton's observation to $\mathbb{R}^4$; he proved, in essence, the isomorphism of the groups Spin_4 and $\mathrm{Sp}_1 \times \mathrm{Sp}_1$. More information on that early period can be found in [9].

Cartan [8] introduced the fundamental representations of the complex Lie algebras $\mathfrak{so}_m$, $m = 3, 4, \dots$, and pointed out that, for $m = 2n$ and $m = 2n + 1$, there are among them irreducible representations of dimension 2^n that do not lift to representations of the orthogonal groups; later they were recognized as the spin representations. The name and fame spinors owe to physicists; it all started with Pauli [19] and Dirac [11]; according to van der Waerden [37] it was Ehrenfest who introduced the name. In Dirac's work, the space of (bi)spinors appears as the carrier of a representation of the Clifford algebra associated with Minkowski space-time. Brauer and Weyl made explicit the connection between Cartan's and Clifford algebra approaches to spinors; they gave a construction, in any number of dimensions, of the spin representations and showed how the tensor product of two such representations decomposes into irreducibles.

For the first time, spinor groups — but not the name — appear in the work of Lipschitz [18]; see remarks on this subject in [13,40].

Shortly after the appearance of Dirac's paper, Wigner, Weyl and Fock developed a (local) formulation of spinor fields, and of their covariant differentiation on Riemannian manifolds. It consisted in referring spinors to tetrads ('Vierbeine'); its modern formulation uses the notion of a (s)pin structure involving a 'prolongation' of the bundle of orthonormal frames to the principal (s)pin bundle; see [A,H,I] for this formulation and [32,34] for further historical remarks on the subject.

Following earlier work by Veblen and Givens [38], Cartan introduced the notion of simple spinors. Chevalley based his *Algebraic theory of spinors* [G] on the notion of minimal ideals of Clifford algebras, an idea considered before by Riesz [27] and, implicitly, by physicists in the context of the Dirac equation [29]. Chevalley developed the theory of Clifford algebras over an arbitrary field of numbers, proved rigorously several fundamental theorems in this subject and introduced the expression *pure spinors* for Cartan's simple spinors. (The adjective 'simple' is reserved in algebra to denote objects that cannot be represented as products.)

3. SPINORS IN PHYSICS

Spinors—the double-connectedness of SO_3—have been 'seen' by physicists in the observation of sodium *doublets*: they indicate the appearance of 2-dimensional vector spaces, carriers of a representation of SU_2. Similar remarks apply to the 'anomalous' Zeeman effect.

One of the great achievements of the 20th century physics is the elucidation of the role of *fermions* in the stability of matter. It is based on Pauli's exclusion principle and the underlying requirement to use Fermi statistics for particles of half-integer spin. The world around us, and life in particular, are so rich because Nature found it convenient to use, among its building blocks, entities requiring spinors in their description.

The Dirac equation led to the prediction of *anti-particles*; the appearance of negative energy states forced quantum field theory to be extended to fermions.

Another major result, of spinorial origin, is the explanation of the value of the *gyromagnetic ratio* of the electron.

Spinors play a fundamental role in supersymmetric and string theories.

There is a wealth of literature on spinors in physics; it suffices here to mention the books [C,D,K,L].

4. THE ALGEBRA OF SPINORS AND NULL ELEMENTS

In this Section, the basic theorems on Clifford algebras and their representations are recalled; their proofs can be found in [C,G,J] and [2].

4.1. Clifford algebras

4.1.1. Definitions.

Definition 1. A *quadratic space* (V, g) is a finite-dimensional vector space V over $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$ with an isomorphism $g : V \to V^*$ which is symmetric, $g^* = g$.

I use the same letter g for the *metric tensor* $g \in V^* \otimes_{\mathrm{sym}} V^*$ associated with that isomorphism so that $g(u,v) = \langle u, g(v) \rangle$. The quadratic form associated with g is denoted by $\boldsymbol{g}$, i.e. $\boldsymbol{g}(v) = g(v,v)$.

Definition 2. The *Clifford algebra* associated with (V,g) is the quotient algebra

$$\mathrm{Cl}(V,g) = \mathrm{Tensor}(V)/\mathrm{I}(V,g),$$

where

$$\mathrm{Tensor}(V) = \bigoplus_{k=0}^{\infty} \otimes^k V, \quad \otimes^0 V = \Bbbk, \quad \otimes^1 V = V, \quad \text{etc.},$$

is the tensor algebra of V and $\mathrm{I}(V,g)$ is the ideal generated by all elements of the form $v \otimes v - \boldsymbol{g}(v)$, $v \in V$.

The Clifford algebra is associative with a unit element denoted by 1. One denotes by κ the canonical map of $\mathrm{Tensor}(V)$ onto $\mathrm{Cl}(V,g)$ and by ab the product of two elements $a, b \in \mathrm{Cl}(V,g)$ so that $\kappa(A \otimes B) = \kappa(A)\kappa(B)$. The map κ is injective on $\Bbbk \oplus V$ and one identifies $\Bbbk \oplus V$ with its image under κ. With this identification, one has

$$uv + vu = 2g(u,v) \quad \text{for every} \quad u, v \in V.$$

4.1.2. The universal property.
Clifford algebras are characterized by their universal property described in

Theorem 1. *If $\mathcal{A}$ is an algebra with a unit $1_{\mathcal{A}}$ and $f : V \to \mathcal{A}$ is a* Clifford map*, i.e. a linear map such that $f(v)^2 = g(v,v)1_{\mathcal{A}}$ for every $v \in V$, then there exists a homomorphism $\hat{f} : \mathrm{Cl}(V,g) \to \mathcal{A}$ of algebras with units such that $f = \hat{f} \circ \kappa | V$.*

Let $\mathcal{A} = \mathrm{End}(\wedge V)$ and, for every $v \in V$ and $w \in \wedge V$, put $f(v)w = v \wedge w + g(v) \lrcorner w$, then $f : V \to \mathrm{End}(\wedge V)$ is a Clifford map and the map $i : \mathrm{Cl}(V,g) \to \wedge V$ given by $i(a) = \hat{f}(a)1_{\wedge V}$ is an isomorphism of vector spaces. This proves

Theorem 2. *As a* vector space*, the algebra $\mathrm{Cl}(V,g)$ is isomorphic to the exterior algebra $\wedge V$; it is $\mathbb{Z}_2$-graded by the* main automorphism *α characterized by $\alpha(v) = -v$ for every $v \in V \subset \mathrm{Cl}(V,g)$.*

Put

$$\mathrm{Cl}^{\pm}(V,g) = \{a \in \mathrm{Cl}(V,g) \mid \alpha(a) = \pm a\}.$$

so that $\mathrm{Cl}(V,g) = \mathrm{Cl}^+(V,g) \oplus \mathrm{Cl}^-(V,g)$ and $\mathrm{Cl}^+(V,g)$ is the even Clifford (sub)algebra. It is convenient to describe the grading by putting $\chi(a) = 0$ for $a \in \mathrm{Cl}^+(V,g)$ and $\chi(a) = 1$ for $a \in \mathrm{Cl}^-(V,g)$. The *transposition* is an antiautomorphism $a \mapsto a^t$ of $\mathrm{Cl}(V,h)$ characterized by being a linear automorphism of the underlying vector space such that $1^t = 1$, $v^t = v$ for every $v \in V$ and $(ab)^t = b^t a^t$ for every $a, b \in \mathrm{Cl}(V,h)$.

If V is m-dimensional, then $\mathrm{Cl}(V,g)$ is 2^m-dimensional. The linear isomorphism

$$c : \mathrm{Cl}(V,g) \to \wedge V \tag{2}$$

defines a $\mathbb{Z}$-grading of the vector space underlying the Clifford algebra: there are elements a_k of $\mathrm{Cl}(V,g)$ of degrees $\deg a_k = k = 0,1,\ldots,m = \dim V$. The Clifford product of two elements of degrees k and l decomposes as follows: $a_k b_l = \sum_{p\in\mathbb{Z}} (a_k b_l)_p$, and [28]

$$(a_k b_l)_p = 0 \quad \text{if} \quad p < |k-l| \quad \text{or} \quad p \equiv k-l+1 \bmod 2 \quad \text{or} \quad p > m - |m-k-l|.$$

One puts $\mathrm{Cl}^k(V,g) = \{a \in \mathrm{Cl}(V,g) \mid \deg a = k\}$. One often uses (2) to *identify* the vector spaces $\bigwedge V$ and $\mathrm{Cl}(V,g)$; this having been done, one can write, for every $v \in V$ and $a \in \mathrm{Cl}(,g)$,

$$va = v \wedge a + g(v) \lrcorner a. \tag{3}$$

4.1.3. The Chevalley theorem

If $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a $\mathbb{Z}_2$-graded algebra and $a \in \mathcal{A}_\varepsilon$, $\varepsilon \in \{0,1\}$, then $\varepsilon = \chi(a)$ is said to be the degree of a. If (V,g) and (W,h) are two quadratic spaces over $\Bbbk$, then their *orthogonal sum* is the quadratic space $(V \oplus W, g \oplus h)$ defined so that $V \perp W$ and $g \oplus h$ restricted to V (resp., to W) is g (resp., h).

Definition 3. Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ be $\mathbb{Z}_2$-graded algebras over $\Bbbk$. Their *graded tensor product* $\mathcal{A} \otimes_{\mathrm{gr}} \mathcal{B}$ is defined so that the its underlying vector space is the tensor product of $\mathcal{A}$ and $\mathcal{B}$, $\chi(a \otimes b) = \chi(a) + \chi(b) \bmod 2$ and

$$(a_1 \otimes b_1) \cdot_{\mathrm{gr}} (a_2 \otimes b_2) = (-1)^{\chi(b_1)\chi(a_2)} a_1 a_2 \otimes b_1 b_2.$$

Note the appearance of a 'supersymmetric' factor in the last equation.

Theorem 3. *Let (V,g) and (W,h) be two quadratic spaces over $\Bbbk$. The algebras*

$$\mathrm{Cl}(V \oplus W, g \oplus h) \quad \textit{and} \quad \mathrm{Cl}(V,g) \otimes_{\mathrm{gr}} \mathrm{Cl}(W,h)$$

are isomorphic.

4.1.4. Structure of Clifford algebras.

Theorem 4. *If the dimension m of V over $\Bbbk$ is even (resp., odd), then the algebra $\mathrm{Cl}(V,g)$ (resp. $\mathrm{Cl}^+(V,g)$), is central simple; as such it has only one, up to equivalence, irreducible and faithful complex representation.*

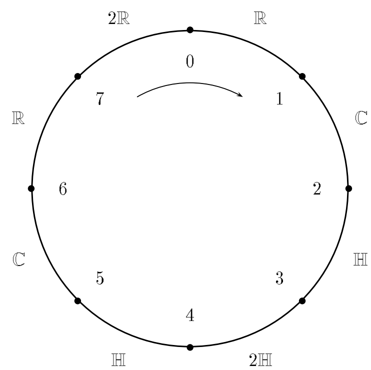
For every set $U \subset V$, one denotes by $U^\perp$ the vector subspace of V consisting of all vectors orthogonal to every element of U.

Theorem 5. *If there is vector $u \in V$ such that $u^2 = -1$—this is always the case when $\Bbbk = \mathbb{C}$—then the map $u^\perp \to \mathrm{Cl}^+(V,g)$ given by $v \mapsto uv$ has the Clifford property and extends to an isomorphism of algebras $\mathrm{Cl}(u^\perp, g|u^\perp) \to \mathrm{Cl}^+(V,g)$.*

The Clifford algebra of the complex vector space $\mathbb{C}^m$ is denoted by Cl_m; the Clifford algebra of the real vector space $\mathbb{R}^m$ with a quadratic form of signature (k,l), $k+l=m$, is denoted by $\mathrm{Cl}_{k,l}$. For any algebra $\mathcal{A}$, one denotes by $2\mathcal{A}$ its double: as a vector space this is $\mathcal{A} \oplus \mathcal{A}$ and multiplication is given by the formula $(a_1,a_2)(b_1,b_2) = (a_1b_1, a_2b_2)$. As a corollary of Theorems 4 and 5 one has

Theorem 6. *The algebras* Cl_{m-1} *and* Cl_m^+ *are isomorphic to each other; the algebra* Cl_{2n} *is isomorphic to the algebra* $\mathbb{C}(2^n)$ *of complex matrices of order* 2^n *and* Cl_{2n}^+ *is isomorphic to the semi-simple algebra* $2\mathbb{C}(2^{n-1})$*. The algebras* $\mathrm{Cl}_{k,l-1}$ *and* $\mathrm{Cl}_{k,l}^+$*,* $l > 0$*, are isomorphic to each other and the complexification* $\mathbb{C} \otimes \mathrm{Cl}_{k,l}$ *is isomorphic to* Cl_{k+l}*. The algebra* $\mathrm{Cl}_{n,n}$ *is isomorphic to* $\mathbb{R}(2^n)$*.*

To describe the structure of all the embeddings $\mathrm{Cl}_{k,l}^+ \to \mathrm{Cl}_{k,l}$ one can use the 'spinorial clock' [E]



To determine $\mathrm{Cl}_{k,l}^0 \to \mathrm{Cl}_{k,l}$, compute the corresponding *hour* $h \in \{0, \dots, 7\}$, $l-k = h+8r$, $r \in \mathbb{Z}$. Read off the sequence $\mathcal{A}_h^0 \xrightarrow{h} \mathcal{A}_h$ from the clock. If $\dim_{\mathbb{R}} \mathcal{A}_h = 2^{\nu_h}$, then $\mathrm{Cl}_{k,l} = \mathcal{A}_h(2^{\frac{1}{2}(k+l-\nu_h)})$, etc. The algebra $\mathcal{A}_h \otimes_{\mathrm{gr}} \mathcal{A}_{h'}$ is of the same type as $\mathcal{A}_{h+h' \bmod 8}$. For a mathematician, the spinorial clock is a way of representing the statement that the graded Brauer group of $\mathbb{R}$ is $\mathbb{Z}_8$ [39].

The expression *Clifford algebra* is used in several meanings. To characterize such an algebra completely, one specifies how the underlying vector space V is embedded in $\mathrm{Cl}(V, g)$. Somewhat less information is given by the injection $\mathrm{Cl}^+(V, g) \to \mathrm{Cl}(V, g)$. Finally, one can treat $\mathrm{Cl}(V, g)$ as an abstract algebra, forgetting where it comes from. For example, as abstract algebras, the algebras $\mathrm{Cl}_{2,0}$ and $\mathrm{Cl}_{1,1}$ are both isomorphic to $\mathbb{R}(2)$, but $\mathrm{Cl}_{2,0}^+ = \mathbb{C}$ whereas $\mathrm{Cl}_{1,1}^+ = 2\mathbb{R}$. In dimension 8, the 3 vector spaces with g of signature $(8, 0)$, $(4, 4)$ and $(0, 8)$ have isomorphic full and even Clifford algebras.

4.1.5. Hodge duality.

Let (e_μ), $\mu = 1, \dots, m$ be an orthonormal frame in (V, g); for $\Bbbk = \mathbb{C}$ it is convenient to take g so that $g(e_\mu, e_\nu) = (-1)^\mu \delta\mu\nu$. Define the volume element associated with (e_μ) as $\eta = e_1 e_2 \dots e_m$. Upon a change of orientation of the frame, the volume element changes sign. One has

$$\eta^2 = 1 \text{ for } \Bbbk = \mathbb{C} \text{ and } \eta^2 = (-1)^{\frac{1}{2}(l-k)(l-k+1)} \text{ for } \Bbbk = \mathbb{R} \text{ and } g \text{ of signature } (k, l).$$

The *Hodge dual* of a multivector a of degree p is the multivector $\star a = a\eta$ of degree $m - p$ and, by virtue of (3),

$$\star(v \wedge a) = g(v) \lrcorner \star a.$$

4.2. Maximal, totally null subspaces of vector spaces

Consider a quadratic space (V, g) over $\Bbbk$. Recall that a vector subspace N of V is said to be *null* if $N^\perp \cap N \neq \varnothing$ and *totally null* if $N \subset N^\perp$. Assume now $V = \mathbb{C}^{2n}$; if $N \subset V$ is *maximal totally null* (*mtn*), then $N^\perp = N$ so that $\dim N = n$. In the complex domain, an orientation having been fixed, the Hodge duality map $\star : \wedge V \to \wedge V$ defined in §4.1.5 satisfies $\star^2 = \mathrm{id}$. If $(m_1, \ldots, m_n)$ is a frame in an *mtn* subspace N, then

$$\star(m_1 \wedge \cdots \wedge m_n) = \pm m_1 \wedge \cdots \wedge m_n. \tag{4}$$

The set of all *mtn* subspaces of $\mathbb{C}^{2n}$ has the structure of a complex manifold, diffeomorphic to the symmetric space $\mathrm{O}_{2n}/\mathrm{U}_n$; its two components correspond to the two signs in (4) characterizing the *mtn* subspaces of positive and negative chiralities, respectively; see, e.g., [25].

Let now (V, g) be a Euclidean quadratic space, i.e. a real quadratic space such that the quadratic form $\boldsymbol{g}$ is positive-definite. Assume that V is of positive even dimension. An *mtn* subspace N of the complexification $W = \mathbb{C} \otimes V$ defines a complex orthogonal structure J on (V, g): this is so because $N \cap \bar{N} = \{0\}$ and one can put

$$J(v) = \mathrm{i}v \quad \text{and} \quad J(\bar{v}) = -\mathrm{i}\bar{v} \quad \text{for} \quad v \in N. \tag{5}$$

Conversely, an orthogonal complex structure J on (V, g) defines the *mtn* subspace $N = \{v \in W | J(v) = \mathrm{i}v\}$.

Consider now a *Lorentz space* (V, g), defined as a real quadratic space such that the quadratic form $\boldsymbol{g}$ is of signature $(2n-1, 1)$, $n = 2, 3, \ldots$. Let $N \subset W = \mathbb{C} \otimes V$ be an *mtn* subspace. The intersection $N \cap \bar{N}$ is the complexification of a null real line $K \subset V$ and

$$N + \bar{N} = \mathbb{C} \otimes K^\perp.$$

There is a real null line L such that $V = K^\perp \oplus L$. The quotient $K^\perp/K$ inherits from (V, g) the structure of a Euclidean quadratic space of dimension $2n - 2$ and there is an orthogonal complex structure J on $K^\perp/K$, defined by

$$J(v \bmod \mathbb{C} \otimes K) = \mathrm{i}v \bmod \mathbb{C} \otimes K \quad \text{for every } v \in \mathbb{C} \otimes K^\perp.$$

4.3. Representations of Clifford algebras

Let again (V, g) be a quadratic space over $\Bbbk$ and let $\mathrm{Cl}(V, g)$ be the corresponding Clifford algebra.

(i) It follows from Theorem 4 that, for $m = 2n$, there is a unique, up to equivalence, faithful and irreducible complex representation

$$\gamma : \mathrm{Cl}(V, g) \to \mathrm{End}\, S \tag{6}$$

of the algebra in the space S of *Dirac spinors* of dimension 2^n. If (e_μ), $\mu = 1, \ldots, 2n$, is an orthonormal frame in V, then $\gamma_\mu = \gamma(e_\mu)$ are the corresponding Dirac matrices and

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}, \quad g_{\mu\nu} = g(e_\mu, \nu).$$

The matrix $\gamma_{2n+1} = \gamma_1\gamma_2 \dots \gamma_{2n}$ anticommutes with all Dirac matrices and its is either the identity $I = \mathrm{id}_S$ or $-I$. One defines Γ to be a matrix anticommuting with all Dirac matrices and such that $\Gamma^2 = I$. The spaces of *Weyl* (chiral, reduced or half) spinors are

$$S_\pm = \{\varphi \in S \mid \Gamma\varphi = \pm\varphi\}$$

so that $S = S_+ \oplus S_-$. The Weyl spinors belonging to S_+ or S_- are said, respectively, to be of positive or negative chirality. The vector space S of Dirac spinors is $\mathbb{Z}_2$-graded: if $\varphi \in S_\pm$, then $\chi(\varphi) \in \{0, 1\}$ is defined by $(-1)^{\chi(\varphi)} = \pm 1$. The representation γ restricted to $\mathrm{Cl}(V, g)$ decomposes into a direct sum of representations γ_+ and γ_- in the spaces S_+ and S_-, respectively. The representation γ is $\mathbb{Z}_2$-graded in the sense that, for $a \in \mathrm{Cl}^\pm(V, g)$ and $\varphi \in S_\pm$, one has $\chi(\gamma(a)\varphi) = \chi(a) + \chi(\varphi) \bmod 2$.

The transposed matrices γ^*_μ, $\mu = 1, \dots, 2n$, define the contragredient representation $\check{\gamma}$ of $\mathrm{Cl}(V, g)$ in the dual space S^*. This representation is equivalent to γ: there is an isomorphism $B : S \to S^*$ such that

$$\gamma^*_\mu = (-1)^n B\gamma_\mu B^{-1} \quad \text{for } \mu = 1, \dots, 2n + 1 \tag{7}$$

and

$$B^* = (-1)^{\frac{1}{2}n(n+1)} B. \tag{8}$$

Denoting by $\sim$ equivalence of representations, it follows from $B\Gamma B^{-1} = (-1)^n\Gamma$ that

$$\check{\gamma}_\pm \sim \gamma_\pm \quad \text{for } n \text{ even} \quad \text{and} \quad \check{\gamma}_\pm \sim \gamma_\mp \quad \text{for } n \text{ odd.}$$

(ii) For $m = 2n + 1$ there are two irreducible, but not necessarily faithful, representations

$$\sigma_\pm : \mathrm{Cl}(V, g) \to \mathrm{End}\, S, \quad \text{such that} \quad \sigma_+ \sim \sigma \circ \alpha,$$

defined in the complex 2^n-dimensional space of *Pauli spinors.* S. The representations σ_+ and σ_- are complex-inequivalent; if $\sigma_+(e_\mu) = \sigma_\mu$, then $\sigma_-(e_\mu) = -\sigma_\mu$. The kernel of σ_- (resp., σ_+) is the vector space of self-dual (resp., antiself-dual) elements of $\mathbb{C} \otimes \mathrm{Cl}(V, g)$. The direct sum $\sigma_+ \oplus \sigma_-$ is a faithful representation of $\mathrm{Cl}(V, g)$ in the space $S_+ \oplus S_-$ of *Cartan spinors*. This representation appears in connection with the Dirac equation on non-orientable, odd-dimensional Riemannian manifolds [31,13]. As in even dimensions, there hold the equivalences

$$\check{\sigma}_\pm \sim \sigma_\pm \quad \text{for } n \text{ even} \quad \text{and} \quad \check{\sigma}_\pm \sim \sigma_\mp \quad \text{for } n \text{ odd.}$$

The restrictions of σ_+ and σ_- to $\mathrm{Cl}^+(V, g)$ are equivalent and give the faithful irreducible *Pauli* representation σ of $\mathrm{Cl}^+(V, g)$ in the 2^n-dimensional complex vector space S of Pauli spinors.

4.3.1. An inductive form of representations of Cl_m.

It is convenient to use a frame in $\mathbb{C}^m$ as in §4.1.5. For every $m = 2n + 1$ one defines a set of anticommuting 'Pauli' matrices $\sigma_1, \dots, \sigma_{2n+1} \in \mathbb{C}(2^n)$ and for every $m = 2n$ one defines a set of anticommuting 'Dirac' matrices $\gamma_1, \dots, \gamma_{2n} \in \mathbb{C}(2^n)$ as follows.

(i) For $m = 1$ put $\sigma_1 = 1$.
(ii) Given the Pauli matrices for $m = 2n - 1$, $n \geqslant 1$, define the Dirac matrices for $m = 2n$ by

$$\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{for } k = 1, \dots, 2n-1 \text{ and } \quad \gamma_{2n} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

where $I \in \mathbb{C}(2^{n-1})$ is the unit matrix.
(iii) Given the Dirac matrices for $m = 2n$, define the Pauli matrices for $m = 2n + 1$ as follows

$$\sigma_k = \gamma_k \quad \text{for } k = 1, \dots, 2n \text{ and } \quad \sigma_{2n+1} = \gamma_1 \gamma_2 \dots \gamma_{2n}.$$

Note that the matrices constructed in this manner have all entries real; they provide a real representation of the real Clifford algebras $\mathrm{Cl}_{n,n-1}$ and $\mathrm{Cl}_{n,n}$ for $n \geqslant 1$. By multiplying by the imaginary unit i the matrices from a suitable subset of the σs or γs, one obtains a set generating an irreducible, complex representation of $\mathrm{Cl}_{k,l}$. For example, for $m = 3$, the matrices $(\gamma_1, \mathrm{i}\gamma_2, \gamma_3, \gamma_4)$ generate a complex representation of $\mathrm{Cl}_{3,1}$. This algebra, however, has a real, faithful and irreducible representation in $\mathbb{R}^4$.

4.3.2. Representations of $\mathrm{Cl}_{k,l}$.

In view of the simplicity of $\mathrm{Cl}(V, g)$ for $\dim V = 2n$, the complex conjugate representation $\bar{\gamma}$ of $\mathrm{Cl}(V, g)$ in the space $\bar{S}$ of complex conjugate spinors is also equivalent to γ: there is an isomorphism $C : S \to \bar{S}$ such that

$$\bar{\gamma}_\mu = C \gamma_\mu C^{-1} \quad \text{for } \mu = 1, \dots, 2n. \tag{9}$$

The spinor

$$\varphi_c = C^{-1} \bar{\varphi} \tag{10}$$

is said to be the *charge conjugate* of $\varphi \in S$. Since $\bar{C}C$ commutes with all the γs, by rescaling, one can achieve either $\bar{C}C = I$ or $\bar{C}C = -I$, depending on the signature of the quadratic form associated with g: $\bar{C}C = I$ for $l - k \equiv 0, 6 \bmod 8$ and $\bar{C}C = -I$ for $l - k \equiv 2, 4 \bmod 8$. Computing $\bar{\Gamma}$ one obtains

$$\bar{\gamma}_\pm \sim \gamma_\pm \quad \text{for } l - k = 0, 4 \text{ and } \quad \bar{\gamma}_\pm \sim \gamma_\mp \quad \text{for } l - k = 2, 6.$$

Similarly, for $m = 2n + 1$ one has

$$\bar{\sigma}_\pm \sim \sigma_\pm \quad \text{for } l - k = 1, 5 \text{ and } \quad \bar{\sigma}_\pm \sim \sigma_\mp \quad \text{for } l - k = 3, 7.$$

The map $A = \bar{B} \circ C : S \to \bar{S}^*$ can be made Hermitian, $A^\dagger = A$, where $A^\dagger = \bar{A}^*$ and $\gamma_\mu^\dagger = (-1)^n A \gamma_\mu A^{-1}$. Incidentally, this notation (A, B and C for the intertwining isomorphisms associated with representations of a Clifford algebra) was introduced, for $m = 4$, by Pauli [20].

The Clifford algebra $\mathrm{Cl}_{k,l}$ exhibits a periodicity of B with respect to $2n = k + l$ and a periodicity of C with respect to $l - k$; they imply a double periodicity, with period 8, of $\mathrm{Cl}_{k,l}$ with respect to both k and l: this motivated us in choosing the title of [E].

4.4. Spinor groups

The spin group $\mathrm{Spin}(V,g)$ is defined as the subset of $\mathrm{Cl}(V,g)$ consisting of Clifford products of all sequences of an even number of unit vectors. The representation γ restricted to $\mathrm{Spin}(V,g)$ decomposes into the direct sum of two complex, inequivalent, irreducible representations γ_+ and γ_- of the spin group in the spaces S_+ and S_-, respectively. Similar remarks apply to the representations γ^* and $\bar{\gamma}$. It is convenient to abuse the language and notation by identifying the carrier spaces with the representations themselves: there thus are the representations $S_+, S_-, S_+^*, S_-^*, \bar{S}_+, \bar{S}_-$, etc. Recall that if ρ is a representation of a group in a complex vector space, then the representations $\rho \oplus \bar{\rho}$ and $\rho \otimes \bar{\rho}$ are both real. In particular, the representations $S \otimes S$, $S_\pm \otimes \bar{S}_\pm$ of $\mathrm{Spin}(V,g)$ are all real; the representations $S_\pm \otimes S_\pm$ are real if $\bar{S}_\pm \sim S_\pm$ and complex otherwise.

The adjoint action of the spin group in $\mathrm{Cl}(V,g)$ also defines a representation of the group; it is, in fact, even a representation of $\mathrm{SO}(V,g)$. It is convenient to complexify this representation; let $W = \mathbb{C} \otimes V$. The complex representation of $\mathrm{Spin}(V,g)$ in $\mathbb{C} \otimes \mathrm{Cl}(V,g) \sim \wedge W$ decomposes according to the $\mathbb{Z}$-grading of $\wedge W$,

$$\wedge W = \oplus_{k=0}^{2n} \wedge^k W, \tag{11}$$

and into the self-dual and anti-self-dual parts,

$$\wedge W = \wedge_+ W \oplus \wedge_- W,$$

where

$$\wedge_\pm W = (I \pm \Gamma) \wedge W.$$

The representations $\wedge^k W$ and $\wedge^{n-k} W$, $k = 0, 1, \ldots, n-1$, are equivalent. For $k \neq n$, the summands on the right of (11) are irreducible and there is the decomposition

$$\wedge^n W = \wedge_+^n W \oplus \wedge_-^n W,$$

into irreducibles.

The representation (6) implies the equivalence

$$S \otimes S^* \sim \wedge W.$$

of complex representations of the spin group. It follows from (7) that

$$S_\pm \sim S_\pm^* \quad \text{for } n \text{ even and} \quad S_\pm \sim S_\mp^* \quad \text{for } n \text{ odd.}$$

If $a \in \wedge^k W$, then $\Gamma a \Gamma^{-1} = (-1)^k a$. Therefore (see §3.3 in [G]),

$$S_\pm \otimes S_\pm \sim \wedge_\pm^n W \oplus \bigoplus_{\substack{k \equiv n \bmod 2 \\ 0 \leqslant k \leqslant n-1}} \wedge^k W, \qquad S_+ \otimes S_- \sim \bigoplus_{\substack{k \equiv n+1 \bmod 2 \\ 0 \leqslant k \leqslant n-1}} \wedge^k W, \tag{12}$$

$$S_\pm \otimes_{\mathrm{sym}} S_\pm \sim \wedge_\pm^n W \oplus \bigoplus_{\substack{k \equiv n \bmod 4 \\ 0 \leqslant k \leqslant n-1}} \wedge^k W \text{ and } \wedge^2 S_\pm \sim \bigoplus_{\substack{k \equiv n+2 \bmod 4 \\ 0 \leqslant k \leqslant n-1}} \wedge^k W. \tag{13}$$

The representation (6) extends to a representation of the complexified Clifford algebra; if $u, v \in V$, then one puts $\gamma(u + \mathrm{i}v) = \gamma(u) + \mathrm{i}\gamma(v)$. If $\varphi \neq 0$, then the vector space

$$N(\varphi) = \{w \in W = \mathbb{C} \otimes V \mid \gamma(w)\varphi = 0\} \tag{14}$$

is totally null.

4.5. Spinor algebra in dimension 4

Spinor calculus in dimension 4 provides an economical, convenient description of many aspects of the geometry of Riemannian manifolds of this dimension. Since there are so many exhaustive presentations of this subject [J,K], it suffices to give here the rudiments of spinor algebra in a form adapted to our purposes.

If the dimension of the real vector space V is 4, then the space of Dirac spinors is also four-dimensional. It follows from (7) and (8) that, in this case, B restricts to a symplectic form on each of the spaces of Weyl spinors S_+ and S_-; following a tradition that goes back to van der Waerden [36], these restrictions are denoted by ε. The representations $S_\pm$ and $S^*_\pm$ are thus equivalent, $S_\pm \sim S^*_\pm$: spinor indices can be lowered by means of ε and raised by means of its inverse. In the van der Waerden-Penrose notation, one labels the components of spinors in S_+ and S_- with letters such as $A, B, \ldots = 1, 2$ and $A', B', \ldots = 1, 2$, and of those in $\bar{S}_+$ and $\bar{S}_-$ by 'dotted indices', $\dot{A}, \dot{B}, \ldots = 1, 2$ and $\dot{A}', \dot{B}', \ldots = 1, 2$, respectively [32]. Thus, if (e^A) is a frame in S^*_+ dual to the 'spin frame' (e_A) in S_+ and $\varphi \in S_+$, then $\varphi = \varphi^A e_A$ and $\varepsilon(\varphi) \in S^*_+$, $\varepsilon(\varphi) = \varepsilon_{AB}\varphi^B e^A$, where $\varepsilon_{AB} = \varepsilon(e_A, e_B)$. One usually uses only unimodular spin frames, i.e. such that $\varepsilon_{12} = 1$. If $\varphi \in S_+$, then $\bar{\varphi} = \bar{\varphi}^{\dot{A}} e_{\dot{A}}$, where $\bar{\varphi}^{\dot{A}} = \overline{\varphi^A}$, if, moreover, $\bar{S}_\pm \sim S_\pm$, then $C(e_A) = C_A{}^{\dot{B}} e_{\dot{B}}$, etc.

It follows now from the first equivalence in (13) that $S_\pm \otimes_{\rm sym} S_\pm \sim \wedge^2_\pm W$. Given $0 \neq \varphi \in S_\pm$, the complex, decomposable bivector corresponding to $\varphi \otimes \varphi$ is associated with $N(\varphi)$ as in (4).

There are three cases to consider, depending on the signature of $\boldsymbol{g}$.

1. If (V, g) is Euclidean, then $\gamma_5^2 = I$, so that one can take $\gamma = \gamma_5$. This implies $C\gamma = \bar{\gamma}C$ and $\bar{S}_\pm \sim S_\pm$ so that there is no need for dotted indices. Since $\bar{C}C = -I$ there are no real ('Majorana') spinors: the representations S and $S_\pm$ are all quaternionic.

Let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

be the Pauli matrices. A convenient representation of the Dirac matrices is

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \ \gamma_4 = \begin{pmatrix} 0 & \mathrm{i}I \\ -\mathrm{i}I & 0 \end{pmatrix},$$

so that

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ C = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \ \text{and} \ A = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{15}$$

Let $0 \neq \varphi \in S_+$ so that $\|\varphi\|^2 = \langle \varphi, \overline{A\varphi} \rangle > 0$. From (12) it follows that

$$\varphi \otimes \overline{A\varphi} = \tfrac{1}{2}\mathrm{i}F + \tfrac{1}{4}(I + \gamma_5)\|\varphi\|^2, \tag{16}$$

where $F = \frac{1}{2}F^{\mu\nu}\gamma_\mu\gamma_\nu$ is a real, self-dual bivector and $F^{\mu\nu}F_{\mu\nu} = \|\varphi\|^4$. The self-duality of F implies the vanishing of the corresponding 'energy-momentum tensor',

$$F^{\mu\sigma}F^{\nu}{}_{\sigma} - \tfrac{1}{4}h^{\mu\nu}F^{\lambda\kappa}F_{\lambda\kappa} = 0.$$

Therefore, the tensor

$$J^{\mu}{}_{\nu} = 2F^{\mu}{}_{\nu}/\|\varphi\|^2 \quad \text{satisfies} \quad J^{\mu}{}_{\sigma}J^{\sigma}{}_{\nu} = -\delta^{\mu}{}_{\nu} \tag{17}$$

and defines a complex structure in V, equivalent to the one defined by the *mtn* subspace $N(\varphi)$.

2. If (V,g) is Lorentzian, then $\gamma_5^2 = -I$, so that $\mathit{\Gamma} = \mathrm{i}\gamma_5$ and $\bar{S}_\pm \sim S_\mp$: dotted indices can be replaced by the primed ones (Penrose's choice). Since $\bar{C}C = I$ there are real Dirac spinors, but, in view of $C\mathit{\Gamma} = -\bar{\mathit{\Gamma}}C$, no real Weyl spinors. In other words, the representation S is real and the representations $S_\pm$ are both complex.

A representation of the Dirac matrices is

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix},\ \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix},\ \gamma_4 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

so that

$$\gamma_5 = \begin{pmatrix} \mathrm{i}I & 0 \\ 0 & -\mathrm{i}I \end{pmatrix},\ C = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

If $0 \neq \varphi \in S_\pm$, then

$$\varphi \otimes \overline{A\varphi} = (I \mp \mathrm{i}\gamma_5)k^\mu \gamma_\mu,$$

where k is a real null vector, spanning the line $N(\varphi) \cap \overline{N(\varphi)}$.

3. If the signature of $\boldsymbol{g}$ is $(2,2)$ ('neutral' or 'split'), then $\gamma_5^2 = I$, so that $C\mathit{\Gamma} = \bar{\mathit{\Gamma}}C$ and $\bar{S}_\pm \sim S_\pm$ as in the Euclidean case. Since now $\bar{C}C = I$, the representations S and $S_\pm$ are all real.

An explicit real representation is given by

$$\gamma_1 = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 0 & \mathrm{i}\sigma_y \\ \mathrm{i}\sigma_y & 0 \end{pmatrix},\ \gamma_3 = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix},\ \gamma_4 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

so that

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\ C = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \text{ and } A = \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}.$$

Let $0 \neq \varphi \in S_+$. There are two cases to consider:

(i) If φ is (proportional to) a real spinor, then $\|\varphi\| = 0$ and $N(\varphi) = \overline{N(\varphi)}$ is the complexification of a real, two-dimensional *mtn* subspace L of V. The bivector F defined as in (16) is the exterior product of two vectors spanning L.

(ii) If $\|\varphi\| \neq 0$, then $N(\varphi) \cap \overline{N(\varphi)} = \{0\}$ and J, defined as in (17), determines a complex structure on V.

4.5.1. The algebraic classification of Weyl tensors

Consider an element φ of the symmetric tensor product $\otimes^r_{\mathrm{sym}} S^*_+$. If $\varphi \neq 0$, then there is a frame (e_A), $A = 1, 2$, in S_+ such that the component $\varphi_{1\dots1} = \varphi(e_1, \dots, e_1)$ is not zero. Given such a frame, let $\psi(z) = ze_1 + e_2 \in S_+$, $z \in \mathbb{C}$, and consider the complex polynomial p_φ of degree r,

$$p_\varphi(z) = \varphi(\psi(z), \dots, \psi(z)) = \varphi_{1\dots1} z^r + \dots + \varphi_{2\dots2}.$$

Let $\{z_1, \dots, z_r\}$ be the set of all roots of this polynomial; a root of multiplicity s appears s times in the set. Then

$$\varphi = \varphi_{1\dots1} \psi^1 \otimes \dots \otimes \psi^r, \quad \text{where} \quad \psi^i_A = \varepsilon_{AB} \psi(z_i)^B, \;\; i = 1, \dots, r.$$

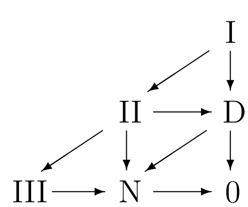
The spinors ψ^i are *eigenspinors* (with eigenvalue 0) of φ. The *algebraic type* of φ is the sequence $[s_1 \dots s_k]$, $1 \leqslant s_1 \leqslant \dots \leqslant s_k \leqslant r$, $s_1 + \dots + s_k = r$, of the multiplicities of the roots of p_φ. In the generic case, all roots are simple, $s_1 = \dots = s_r = 1$. Otherwise, one says that φ is *algebraically degenerate*.

For even $r = 2l$, the $(2l+1)$-dimensional representations $S^{2l}_\pm = \otimes^{2l}_{\mathrm{sym}} S^*_\pm$ are tensorial: they are representations of $\mathrm{O}_4(\mathbb{C})$. For example, for $r = 2$, every algebraically degenerate element of $S^*_+ \otimes_{\mathrm{sym}} S^*_+$ is of the form $\psi \otimes \psi$ for some $\psi \in S^*_+$: it corresponds to a self-dual, decomposable bivector. If $\bar{S}_\pm \sim S_\pm$, then the representations $S^{2l}_\pm$ are real. For example, the reality condition for $\varphi \in S^2_+$ is

$$C_A{}^{\dot{A}} C_B{}^{\dot{B}} \bar{\varphi}_{\dot{A}\dot{B}} = \varphi_{AB}. \tag{18}$$

The spaces $\otimes^4_{\mathrm{sym}} S^*_+$ and $\otimes^4_{\mathrm{sym}} S^*_-$ are isomorphic to spaces of tensors of rank 4 over $W = \mathbb{C}^4$, with symmetries of self-dual and anti-self-dual Weyl (conformal curvature) tensors, denoted by C_+ and C_-, respectively. The enumeration of the possible degeneracies can be traced back to Cartan [9]; physicists use it now in a form due to Penrose [21]:

(i) Type I (non-degenerate) [1111],
(ii) Type II [112],
(iii) Type III [13],
(iv) Type D ('degenerate') [22],
(v) Type N ('null') [4].



The 0 in the Penrose diagram above represents a vanishing φ. The arrows point towards more special cases. This classification of complex, self-dual Weyl tensors is often associated with the name of Petrov, who, however, recognized only three types (I, II and III). The Weyl tensor of a complex Riemannian manifold decomposes into its self-dual and anti-self-dual parts; their algebraic types are independent.

In the case of real manifolds, one has to consider separately each signature. I restrict myself to the proper Riemannian and Lorentz cases.

1. In the proper Riemannian case, the Weyl tensor decomposes into the real, self-dual and anti-self-dual parts; they are independent. The self-dual part is represented by a spinor $\varphi \in S^4_+$ that satisfies a reality condition analogous to (18); in view of (15) this implies the equality

$$\overline{p_\varphi(z)} = \bar{z}^4 p_\varphi(-1/\bar{z})$$

which shows that the eigenspinors of φ occur in pairs (ψ, ψ_c), where ψ_c is the charge conjugate of $\psi \in S_+^*$. Therefore, there are only two types of $\varphi \neq 0$: either these two pairs are distinct (type I) or they coincide (type D). Similar remarks apply to the anti-self-dual part of the Weyl tensor. Therefore, the complete algebraic classification of the Weyl tensor of a proper Riemannian 4-dimensional manifold contains 9 cases; (I,I) is the most general case and (0,0) represents conformally flat manifolds. The cases $(*,0)$ and $(0,*)$ are referred to as self-dual and anti-self-dual, respectively.

2. In the Lorentzian case, the real Weyl tensor decomposes into its self- and anti-self-dual parts, which are complex, $\mathsf{C} = \mathsf{C}_+ + \mathsf{C}_-$, where $\star\mathsf{C}_\pm = \pm\mathrm{i}\mathsf{C}_\pm$ so that $\bar{\mathsf{C}}_+ = \mathsf{C}_-$. Therefore, the classification is given by that of the complex, self-dual Weyl tensor presented above.

4.6. Multivectors associated with pairs of spinors

The tensor product of two spin representations of $\mathrm{Spin}(V,h)$ is a representation of $\mathrm{SO}(V,h)$; this simple fact underlies the physicists' construction of (real) multivectors from spinors.

4.6.1. The complex case.

m odd

Consider first an *odd*-dimensional complex vector space $W = \mathbb{C}^{2n+1}$ $(n = 1, 2, \dots)$ and the corresponding Pauli representation $\sigma : \mathrm{Cl}_{2n+1}^+ \to \mathrm{End}\, S$; see §4.3.

The representations $\sigma_\pm$ can be described explicitly as follows. Consider a *Witt decomposition* $W = N \oplus P \oplus \mathbb{C}e_{2n+1}$, where N and P are n-dimensional—therefore maximal—totally null subspaces of W and e_{2n+1} is a unit vector orthogonal to $V = N \oplus P$. One takes $S = \wedge N$ and, writing an element of W as $n + p + ze_{2n+1}$, where $n \in N$, $p \in P$ and $z \in \mathbb{C}$, one puts $\sigma_\pm(n + p + ze_{2n+1}) = \pm(\sqrt{2}(e(n) + i(p)) + z\alpha_N)$.

Let β be the antiautomorphism of Cl_{2n+1} defined by $\beta(a) = a^t$ for n even and $\beta(a) = \alpha(a)^t$ for n odd so that $\beta(\eta) = \eta$ for every n. The two representations $a \mapsto \sigma_\pm(\beta(a))^*$ of Cl_{2n+1} in S^* are equivalent to the corresponding representations $\sigma_\pm$: there exists an isomorphism $B : S \to S^*$ such that $\sigma_\pm(\beta(a))^* = B\sigma_\pm(a)B^{-1}$ for every $a \in \mathrm{Cl}_{2n+1}$. Iterating and using Schur's lemma one obtains $B^* = \varepsilon B$, where either $\varepsilon = 1$ or $\varepsilon = -1$. To determine ε, note that $(B\sigma(a))^* = \varepsilon B\sigma(a^t)$ for every $a \in \mathrm{Cl}_{2n+1}^+$. Since $\dim\{f : S \to S^* \mid f^* = f\} > \dim\{f : S \to S^* \mid f^* = -f\}$, one has $\varepsilon = \mathrm{sgn}\,(\dim A_n^+ - \dim A_n^-)$, where $A_n^\pm = \{a \in \mathrm{Cl}_{2n+1}^+ \mid a^t = \pm a\}$. Moreover, $\dim A_n^+ - \dim A_n^- = \sum_{p=0}^n (-1)^p \binom{2n+1}{2p} = 2^n\sqrt{2}\cos(2n+1)\frac{\pi}{4}$; this gives (8). For every $a \in \mathrm{Cl}_{2n+1}^+$ one has

$$\sigma(a)^* = B\sigma(a^t)B^{-1}. \tag{19}$$

The isomorphism B defines a non-degenerate quadratic forms $B \otimes B^{-1}$ and $B \otimes B$ on $\mathrm{End}\, S$ and $S \otimes S$, respectively. Namely, if $f \in \mathrm{End}\, S$, then $(B \otimes B^{-1})(f) = \mathrm{tr}(B^{-1} \circ f^* \circ B \circ f)$; if $\varphi, \psi \in S$, then $(B \otimes B)(\varphi \otimes \psi) = \langle \varphi, B\varphi \rangle \langle \psi, B\psi \rangle$; the linear map $S \otimes S \to \mathrm{End}\, S$ defined by $\varphi \otimes \psi \mapsto \varphi \otimes B\psi$ is an isometry for these quadratic forms. Similarly, the algebra Cl_{2n+1}^+ has a quadratic form H, $H(a) = 2^{-n}\mathrm{tr}\,\sigma(a^t a)$ and σ is an isometry of (Cl_{2n+1}, H) onto $(\mathrm{End}\, S, 2^{-n}B \otimes B^{-1})$. The even exterior algebra $\wedge^+ W$ has a quadratic form $\wedge^+\boldsymbol{g}$ obtained by extension of $\boldsymbol{g}$; the isomorphism (2) restricted to Cl_{2n+1}^+ is an isometry equivariant with respect to the action of the spin group. This leads to (see

Prop. 4. 2. in [E] and Prop. 2 in [32])

Proposition 1. *Let* $\sigma : \mathrm{Cl}^+_{2n+1} \to S$ *be the Pauli representation. There then exists an isomorphism* $B : S \to S^*$ *such that* (8) *and* (19) *hold. The bilinear map*

$$E : S \times S \to \wedge^+ W, \quad E(\varphi, \psi) = c \circ \sigma^{-1}(\varphi \otimes B\psi),$$

(i) *satisfies*

$$E(\psi, \varphi) = (-1)^{\frac{1}{2}n(n+1)} E(\varphi, \psi)^t;$$

(ii) *if* $v \in W$, *then*

$$E(\sigma_+(v)\varphi, \psi) = (i(v) + e(v)) \star E(\varphi, \psi);$$

(iii) *if* $a \in \mathrm{Spin}_{2n+1}(\mathbb{C})$, *then*

$$E(\sigma(a)\varphi, \sigma(a)\psi) = \wedge\rho(a) \circ E(\varphi, \psi);$$

(iv) *the linear map* $S \otimes S \to \wedge^+ W$, *associated with* E, *is an isometry of the quadratic space* $(S \otimes S,\ 2^{-n} B \otimes B)$ *onto* $(\wedge^+ W,\ \wedge^+ h)$ *which is equivariant with respect to the action of the group* $\mathrm{Spin}_{2n+1}(\mathbb{C})$.

The component of E in $\wedge^{2p} W$ is denoted by E_{2p}. In particular, $E_0(\varphi, \psi) = 2^{-n}\langle \varphi, B\psi \rangle$. The bilinear form E_0 is invariant with respect to the action of $\mathrm{Spin}_{2n+1}(\mathbb{C})$. More generally, if $a \in \mathrm{Cl}^+_{2n+1}$ and $a^t a = 1$, then $\langle \sigma(a)\varphi, B\sigma(a)\psi \rangle = \langle \varphi, B\psi \rangle$. According to part (i) of Prop. 1, one has

$$E_{2p}(\psi, \varphi) = (-1)^{\frac{1}{2}n(n+1)+p} E_{2p}(\varphi, \psi).$$

Putting ν equal to the integer part of $\frac{1}{2}(n+1)$, one obtains that

$$\text{if } \nu - p \text{ is odd, then } E_{2p}(\varphi, \varphi) = 0. \tag{20}$$

m even

Consider now the $2n$-dimensional subspace V of $W = \mathbb{C}^{2n+1}$ orthogonal to the unit vector e_{2n+1}. The algebra Cl_{2n} can be identified with a subalgebra of Cl_{2n+1}; the Clifford map $V \to \mathrm{Cl}^+_{2n+1}$, $v \mapsto \eta v$, extends to an isomorphism of algebras $j : \mathrm{Cl}_{2n} \to \mathrm{Cl}^+_{2n+1}$. Since $j(v)^t = (-1)^m j(v)$, one has $j(a)^t = j(\beta(a))$ for every $a \in \mathrm{Cl}_{2n}$. The element $\eta e_{2n+1} = e_1 \cdots e_{2n}$ is a volume in V. The composition $\gamma = \sigma \circ j$ is the 'Dirac' representation of the algebra Cl_{2n} in S. One has $\sigma_\pm(e_\mu) = \pm\gamma_\mu$ for $\mu = 1, \ldots, 2m$ and $\sigma_\pm(e_{2m+1}) = \pm\Gamma$. These definitions imply (7) and (8).

Let $k : \wedge V \to \wedge^+ W$ be the isomorphism of vector spaces such that $k \circ c = c \circ j$; explicitly, it is given by $k(w) = w$ for $w \in \wedge^+ V$ and $k(w) = \star w$ for w odd, $w \in \wedge^- V$. Let $*$ denote the Hodge dual in V so that $c(\eta e_{2n+1} a) = *c(a)$ for $a \in \mathrm{Cl}_{2n}$. Consider the bilinear map

$$F = k^{-1} \circ E : S \times S \to \wedge V.$$

Denoting by $F_p(\varphi, \psi)$ the component of $F(\varphi, \psi)$ in $\wedge^p V$, one obtains, as a corollary of part (i) of Prop. 1,

$$F_p(\psi, \varphi) = (-1)^{\frac{1}{2}(n-p)(n-p+1)} F_p(\varphi, \psi). \tag{21}$$

Putting $v = e_{2n+1}$ in part (ii) and using $\sigma_+(e_{2n+1}) = \sigma(\eta e_{2n+1})$ leads to

$$F(\Gamma\varphi, \psi) = *F(\varphi, \psi) \quad \text{and} \quad F(\varphi, \Gamma\psi) = (-1)^n \alpha \circ *F(\varphi, \psi). \tag{22}$$

If $\wedge V$ is given the quadratic form $\wedge \boldsymbol{g}$, then k becomes an isometry of $(\wedge V, \wedge \boldsymbol{g})$ onto $(\wedge^+ W, \wedge^+ \boldsymbol{g})$. As a corollary from Prop. 1 one obtains that the map

$$S \otimes S \to \wedge V, \text{ given by } \varphi \otimes \psi \mapsto c \circ \gamma^{-1}(\varphi \otimes B\psi),$$

is an isometry of the corresponding quadratic spaces, equivariant with respect to the action of $\mathrm{Spin}_{2n}(\mathbb{C})$.

Recall that γ restricted to the Cl^+_{2n} decomposes into the direct sum of the Weyl representations γ_+ and γ_- in s S_+ and S_-. If φ and ψ are both Weyl spinors with respect to γ, then (7) and (21) give

$$\text{if} \quad \chi(\varphi) + \chi(\psi) + n + p \equiv 1 \mod 2, \quad \text{then} \quad F_p(\varphi, \psi) = 0.$$

In particular, if φ is a Weyl spinor, then $F_p(\varphi, \varphi) = 0$ unless $p \equiv n \mod 4$.

If the representation γ comes from a representation σ constructed in terms of a Witt decomposition, so that $S = \wedge N$, then $\Gamma = \alpha_N$ and $S_\pm = \wedge^\pm N$.

4.6.2. The real case.

Consider now the real vector space $\mathbb{R}^m$ with a quadratic form $\boldsymbol{g}$ of signature (k, l), $k+l = m$. If $(e_1, \ldots, e_m)$ is a frame orthonormal with respect to g, then the volume element $\eta = e_1 \cdots e_n$ satisfies $\eta^2 = (-1)^{\frac{1}{2}(l-k)(l-k+1)}$. The complexification of the Clifford algebra $\mathrm{Cl}_{k,l}$ is isomorphic with Cl_m.

***m* odd**

Let $m = 2n+1$. The Pauli representation of Cl^+_{2n+1}, restricted to $\mathrm{Cl}^+_{k,l}$, yields a representation σ of this real algebra in a 2^n-dimensional, complex vector space S. The representation σ can be extended to the representations σ_+ and σ_- of $\mathrm{Cl}_{k,l}$ in S by putting $\sigma_\pm(\eta) = \pm \mathrm{id}_S$ when $\eta^2 = 1$ and $\sigma_\pm(\eta) = \pm \mathrm{i}\,\mathrm{id}_S$ when $\eta^2 = -1$.

With every representation τ of a *real* algebra $\mathcal{A}$ in a *complex* vector space S one can associate the complex conjugate representation $\bar\tau$ in $\bar S$, given by $\bar\tau(a) = \overline{\tau(a)}$ for every $a \in \mathcal{A}$. Since $\mathrm{Cl}^+_{k,l}$ is central simple for $k+l$ odd, its representations σ and $\bar\sigma$ are complex-equivalent; if $\eta^2 = 1$, then the representations $\overline{\sigma_\pm}$ are equivalent to the corresponding representations $\sigma_\pm$; if $\eta^2 = -1$, then $\overline{\sigma_+}$ is equivalent to σ_-. In every case there is a linear isomorphism

$$C : S \to \overline{S} \text{ such that } \overline{\sigma_\mu} = (-1)^{\frac{1}{2}(l-k)(l-k+1)} C \sigma_\mu C^{-1}, \text{ where } \sigma_\mu = \sigma_+(e_\mu)$$

for $\mu = 1, \ldots, k+l = 2n+1$. An argument similar to the one used with respect to B in §4.6.1 shows that C can be rescaled so that either $\bar C C = \mathrm{id}_S$ or $\bar C C = -\mathrm{id}_S$. Moreover, one obtains from (19) and

$$\overline{\sigma(a)} = C\sigma(a)C^{-1}, \quad a \in \mathrm{Cl}^+_{k,l},$$

that $C^{-1}\bar B^{-1}\bar C^* B^*$ is in the commutant of σ; therefore, one can rescale B so that

$$B = C^* \bar B C$$

and then the sesquilinear form

$$A : S \times S \to \mathbb{C}, \quad \text{given by} \quad A(\varphi, \psi) = \langle \bar{\varphi}, \bar{B}C\psi \rangle$$

is either Hermitean or anti-Hermitean.

The charge conjugate φ'_c of $\varphi' \in S^*$ is defined so that $\langle \varphi_c, \varphi'_c \rangle = \overline{\langle \varphi, \varphi' \rangle}$; by virtue of (14), if $\varphi' = B\psi$, then $\varphi'_c = B\psi_c$. There are two cases to consider:
(i) The *real* case: if $l - k \equiv 1$ or $7 \mod 8$, then $\bar{C}C = \mathrm{id}_S$; there is then the 2^n-dimensional real vector space

$$S_\mathbb{R} = \{\varphi \in S \mid \varphi_c = \varphi\}.$$

and a decomposition of S into complementary subspaces of 'Majorana' spinors, $S = S_\mathbb{R} \oplus$ i@, $S_\mathbb{R}$. The representation σ is real: $\sigma(a)S_\mathbb{R} \subset S_\mathbb{R}$ for every $a \in \mathrm{Cl}^+_{k,l}$. The automorphisms $\sigma_\mu = \sigma_+(e_\mu)$ are real (resp., pure imaginary) for $l - k \equiv 7 \mod 8$ (resp., $l - k \equiv 1 \mod 8$). The algebra $\mathrm{Cl}^+_{k,l}$ is isomorphic to the matrix algebra $\mathbb{R}(2^n)$. The algebra $\mathrm{Cl}_{k,l}$ is isomorphic to $\mathbb{R}(2^n) \oplus \mathbb{R}(2^n)$ (resp., $\mathbb{C}(2^n)$) for $l - k \equiv 7 \mod 8$ (resp., $l - k \equiv 1 \mod 8$). The form A restricted $S_\mathbb{R}$ is real and has the same symmetry as B.
(ii) The *quaternionic* case: if $l - k \equiv 3$ or $5 \mod 8$, then $\bar{C}C = -\mathrm{id}_S$ and S can be given the structure of a right module over $\mathbb{H}$. Explicitly, denoting by i, j and $\mathrm{k} = \mathrm{ij}$ the quaternionic units, one puts $\varphi \mathrm{i} = \sqrt{-1}\,\varphi$ and $\varphi \mathrm{j} = \varphi_c$. The algebra $\mathrm{Cl}^+_{k,l}$ is isomorphic to the matrix algebra $\mathbb{H}(2^{n-1})$. The algebra $\mathrm{Cl}_{k,l}$ is isomorphic to $2\mathbb{H}(2^{n-1})$ (resp., $\mathbb{C}(2^m)$) for $l - k \equiv 3 \mod 8$ (resp., $l - k \equiv 5 \mod 8$).

The vector space $\mathcal{A} = \{f \in \mathrm{End}\, S \mid \bar{f}C = Cf\}$ is a real algebra spanned by all elements of the form $\varphi \otimes \varphi' + \varphi_c \otimes \varphi'_c$, where $\varphi \in S$ and $\varphi' \in S^*$. The representation σ factors through the injection $\mathcal{A} \to \mathrm{End}\, S$. Moreover, in the real case, the algebra $\mathcal{A}$ is isomorphic to $S_\mathbb{R} \otimes_\mathbb{R} S_\mathbb{R}$. In the quaternionic case, it is isomorphic to the tensor product over $\mathbb{H}$ of the right $\mathbb{H}$-module S by the left $\mathbb{H}$-module S^*. In each case one has

$$\overline{E(\varphi, \psi)} = E(\varphi_c, \psi_c).$$

The homogeneous components of the multivector $E(\varphi_c, \varphi)$ are either real or imaginary, as can be seen from part (i) of Prop. 1 and $\varphi_{cc} = C\bar{C}\varphi$.

The quadratic form H on $\mathrm{Cl}^+_{k,l}$, $k + l = 2n + 1$, $H(a) = 2^{-n}\mathrm{tr}\,\sigma(a^t a)$, is real; its signature can be evaluated as follows. Consider the polynomial $\varsigma(\xi, \eta) = \frac{1}{2}(1 + \xi)^k(1 + \eta)^l + \frac{1}{2}(1 - \xi)^k(1 - \eta)^l = \sum_{p,q;\, p+q \text{ even}} \binom{k}{p}\binom{l}{q}\xi^p\eta^q$. The index of H equals $\varsigma(1, -1)$. Therefore, H is positive-definite if, and only if, either $k = 0$ or $l = 0$; if both k and l are positive, then H is neutral.

m even

Consider now the *even*-dimensional subspace V of $\mathbb{R}^{k+l}$, orthogonal to a unit vector u. Depending on whether $u^2 = 1$ or -1, the signature of the restriction $\boldsymbol{g}_V$ of $\boldsymbol{g}$ to V is $(k - 1, l)$ or $(k, l - 1)$. The map $V \to \mathrm{Cl}^+_{k,l}$, $v \mapsto v\eta$, extends to an isomorphism of algebras $j : \mathrm{Cl}(V, \eta^2 h_V) \to \mathrm{Cl}^+_{k,l}$: if $\eta^2 = 1$ (resp., $\eta^2 = -1$), then $\mathrm{Cl}^+_{k,l}$ is isomorphic to both $\mathrm{Cl}_{k-1,l}$ and $\mathrm{Cl}_{k,l-1}$ (resp., $\mathrm{Cl}_{l-1,k}$ and $\mathrm{Cl}_{l,k-1}$).

If $k + l = 2n$ and $\gamma : \mathrm{Cl}_{k,l} \to \mathrm{End}\, S$ is a Dirac representation in a complex space S of dimension 2^n, then there is $C : S \to \bar{S}$ such that $\overline{\gamma(a)} = C\gamma(a)C^{-1}$ for every $a \in \mathrm{Cl}_{k,l}$. According to previous remarks, one can rescale C so that either $\bar{C}C = \mathrm{id}_S$ (for $l - k \equiv 0$

or 6 mod 8) or $\bar{C}C = -\mathrm{id}_S$ (for $l-k \equiv 2$ or 4 mod 8). It is convenient to define the chirality automorphism as $\Gamma = (-1)^{\frac{1}{4}(l-k)(l-k+1)}\gamma_1 \cdots \gamma_{2m}$ so that

$$\Gamma^2 = \mathrm{id}_S \quad \text{and} \quad \bar{\Gamma} = (-1)^{\frac{1}{2}(l-k)(l-k+1)} C\Gamma C^{-1}.$$

The representation γ, restricted to $\mathrm{Cl}^+_{k,l}$ decomposes as in the complex case, $\gamma = \gamma_+ \oplus \gamma_-$. The representation $\check{\gamma}$ of $\mathrm{Cl}^+_{k,l}$ in S^*, contragredient to γ, is defined by $\check{\gamma}(a) = \gamma(a^t)^*$, $a \in \mathrm{Cl}^+_{k,l}$. It also decomposes, $\check{\gamma} = \check{\gamma}_+ \oplus \check{\gamma}_-$, where $\check{\gamma}_\pm : \mathrm{Cl}^+_{k,l} \to \mathrm{End}\, S^*_\pm$, $S^*_\pm = \{\varphi \in S \mid \Gamma^*\varphi = \pm\varphi\}$. There is also a similar decomposition $\bar{\gamma} = \bar{\gamma}_+ \oplus \bar{\gamma}_-$ of $\bar{\gamma}$ restricted to $\mathrm{Cl}^+_{k,l}$. The representations γ_+ and γ_- are not complex-equivalent; each of the representations $\check{\gamma}_+$, $\check{\gamma}_-$, $\bar{\gamma}_+$ and $\bar{\gamma}_-$ is equivalent to either γ_+ or γ_-; see §4.3.

5. Pure spinors

Consider a non-zero spinor $\varphi \in S$ associated with $W = \mathbb{C}^{2n+1}$. The vector space

$$N(\varphi) = \{v \in W \mid \sigma(\eta v)\varphi = 0\}$$

is totally null; its dimension is called the *nullity* of φ. A spinor $\varphi \neq 0$ is said to be *pure* if its nullity is maximal, i. e. equal to n. Let $W = N \oplus P \oplus \mathbb{C}u$ be a Witt decomposition and let $(n_1, \ldots, n_n)$ be a basis of N. Put $a_N = n_1 \cdots n_n$ for n even and $a_N = \eta n_1 \cdots n_n$ for n odd, so that $a_N \in \mathrm{Cl}^+_{2n+1}$. Since σ is faithful, there is a spinor φ_0 such that $\varphi = \sigma(a_N)\varphi_0 \neq 0$ and then $N(\varphi) = N$ so that φ is pure. If ψ is another spinor such that $N(\psi) = N$, then there is $z \in \mathbb{C}^\times$ such that $\psi = z\varphi$: there is a bijective correspondence between the set (in fact, a compact, connected, complex manifold Σ_{2n+1} of complex dimension $\frac{1}{2}n(n+1)$) of directions of pure spinors and the set of maximal null subspaces of W. If $a \in \mathrm{Spin}_{2n+1}(\mathbb{C})$ and φ is pure, then $\sigma(a)\varphi$ is also pure; the induced action of $\mathrm{Spin}_{2n+1}(\mathbb{C})$ on Σ_{2n+1} is transitive.

Let $\varphi \neq 0$ be a spinor of nullity q. Let $(n_1, \ldots, n_q)$ be a basis of $N(\varphi)$. It follows from part (ii) of Prop. 1 and (20) that $n \in N(\varphi)$ implies $e(n)E_{2p}(\varphi,\varphi) = 0$ and $i(n)E_{2p}(\varphi,\varphi) = 0$ for every p such that $p \equiv \nu \bmod 2$. Therefore, there is a $(2p-q)$-vector E'_{2p-q} such that $E_{2p}(\varphi,\varphi) = n_1 \wedge \cdots \wedge n_q \wedge E'_{2p-q}$ and $i(n)E'_{2p-q} = 0$ for every $n \in N(\varphi)$. This implies that if $2p < q$ or $2p > 2m+1-q$, then $E_{2p}(\varphi,\varphi) = 0$. In particular, if φ is pure, then $E_{2p}(\varphi,\varphi) \neq 0$ if, and only if, $p = \nu$. A more precise result is contained in

Theorem 7. *A spinor $\varphi \neq 0$ is pure if, and only if, $p \neq \nu$ implies $E_{2p}(\varphi,\varphi) = 0$. If φ is pure, then there is a basis $(n_1, \ldots, n_n)$ of $N(\varphi)$ such that*

$$\textit{if } n \textit{ is even, then} \quad E_n(\varphi,\varphi) = n_1 \wedge \cdots \wedge n_n;$$

$$\textit{if } n \textit{ is odd, then} \quad \star E_{n+1}(\varphi,\varphi) = n_1 \wedge \cdots \wedge n_n.$$

The somewhat difficult proof of the 'if' part of Theorem 7 appears in [F,G]. By considering the multivectors $E_{2p}(\varphi,\varphi)$ for low values of n, one obtains, as a corollary of (20) and Theorem 7 that all spinors associated with W of dimension 3 and 5 are pure; in dimensions 7 and 9 pure spinors belong to the cone of equation $E_0(\varphi,\varphi) = 0$.

If φ is pure and $v \in W$ is non-null, $v^2 \neq 0$, then $\sigma(\eta v)\varphi$ is also pure: $N(\sigma(\eta v)\varphi) = vN(\varphi)v^{-1}$. In particular, if $N(\varphi)$ is orthogonal to the unit vector e_{2n+1}, then

$$N(\sigma(\eta e_{2n+1})\varphi) = N(\varphi).$$

Therefore, $\sigma(\eta e_{2n+1})\varphi = \pm\varphi$. Introducing, as in part (ii) of §4.6.1, the $2n$-dimensional space V orthogonal to e_{2n+1}, one obtains, as a corollary of Theorem 7 and (22):

Theorem 8. *Let $W = V \oplus \mathbb{C}e_{2n+1}$ and γ be the Dirac representation of Cl_{2n} in S, $\gamma(v) = \sigma(\eta v)$ for $v \in V$. If φ is pure, then $N(\varphi) \subset V$ if, and only if, φ is a Weyl spinor with respect to γ. Assuming that φ is such a spinor, one has $F_p(\varphi,\varphi) = 0$ for every $p \neq n$ and there is a basis $(n_1, \ldots, n_n)$ in $N(\varphi)$ so that*

$$F_n(\varphi,\varphi) = n_1 \wedge \cdots \wedge n_n.$$

The n-vector $F_n(\varphi,\varphi)$ is either self-dual ($\Gamma\varphi = \varphi$) or antiself-dual ($\Gamma\varphi = -\varphi$).

In other words, in even-dimensional complex vector spaces, tensor squares of pure spinors define self- or antiself-dual decomposable multivectors of the middle degree. A pure spinor φ associated with the representation $\gamma : \mathrm{Cl}_{2n} \to \mathrm{End}\, S$ can be characterized, without reference to the odd-dimensional space W, by $\dim\{v \in \mathbb{C}^{2n} \mid \gamma(v)\varphi = 0\} = n$; it follows that it is a Weyl spinor. The set of directions of pure spinors associated with γ is a $\frac{1}{2}n(n-1)$-dimensional complex compact manifold $\Sigma_{2n} = \mathrm{O}_{2n}/\mathrm{U}_n$; it has two connected components, Σ^+_{2n} and Σ^-_{2n}, corresponding to pure spinors of opposite chiralities. An argument similar to the one used in odd dimensions shows that, in dimensions 2, 4 and 6 all Weyl spinors are pure. In dimension 8 pure spinors lie on the cones of equation $F_0(\varphi,\varphi) = 0$ in S_+ and S_-; in dimension 10 pure spinors are characterized by the equation $F_1(\varphi,\varphi) = 0$ in $S_\pm$ and generic Weyl spinors have nullity 1; for $n = 4$ and $n > 5$ generic Weyl spinors have zero nullity. Spinors belonging to one orbit of the group $\mathrm{Spin}_{2n}(\mathbb{C})$ have the same nullity, but the converse is not true: in dimensions $\geqslant 12$ nullity of Weyl spinors provides a rather coarse classification of the orbits. There are no Weyl spinors of nullity q such that $n - 4 < q < n$. All homogeneous polynomial invariants of the spin group vanish on spinors of positive nullity [35].

6. THE CALCULUS OF SPINORS

6.1. Covariant differentiation of spinor fields

This section is adapted from [34]. To present the notion of covariant differentiation of spinor fields in a language familiar to physicists it is convenient to use the terminology of *gauge fields*. For simplicity, consider an even-dimensional manifold, $k + l = 2n$, put $G = \mathrm{Pin}_{k,l}$ and let a representation of $\mathrm{Cl}_{k,l}$ in S be given by the Dirac matrices $\gamma_\mu \in \mathrm{End}\, S$. A spinor field is now a map $\psi : M \to S$; given a function $U : M \to G$, one defines the gauge-transformed spinor field as $\psi' = U^{-1}\psi$, $\psi'(x) = U(x)^{-1}\psi(x)$ for $x \in M$. A spinor connection ('gauge potential') is a 1-form ω on M with values in the Lie algebra of G, i.e. in $\mathrm{Cl}^2_{k,l} \subset \mathrm{End}\, S$; therefore, it can be written as $\omega = \frac{1}{4}\gamma^\mu\gamma^\nu\omega_{\mu\nu}$, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are 1-forms. The covariant ('gauge') derivative of ψ is

$$D\psi = \mathrm{d}\psi + \omega\psi. \tag{23}$$

The gauge transformation U induces a change of the connection, $\omega \mapsto \omega' = U^{-1}\omega U + U^{-1}\mathrm{d}U$ so that $(D\psi)' = U^{-1}D\psi$. Since the dimension of M is even, the adjoint representation is onto $\mathrm{O}_{k,l}$ and one can define, for every $a \in \mathrm{Pin}_{k,l} \subset \mathrm{GL}(S)$, the (orthogonal) matrix $(\rho^\mu{}_\nu(a))$ by $a^{-1}\gamma^\mu a = \rho^\mu{}_\nu(a)\gamma^\nu$, so that $a^{-1}\gamma_\mu a = \gamma_\nu \rho^\nu{}_\mu(a^{-1})$. From the Lemma: if $a \in \mathrm{Cl}^p_{k,l}$, then $g^{\mu\nu}\gamma_\mu a \gamma_\nu = (-1)^p(n-2p)a$, taking into account that $U^{-1}\mathrm{d}U$ is in the Lie algebra of G—therefore of degree $p=2$—one obtains $g^{\mu\nu}U^{-1}\gamma_\mu U \mathrm{d}(U^{-1}\gamma_\nu U) = 4U^{-1}\mathrm{d}U$ so that $\omega'^\mu{}_\nu = \rho^\mu{}_k(U^{-1})\omega^k{}_l\rho^l{}_\nu(U) + \rho^\mu{}_k(U^{-1})\mathrm{d}\rho^k{}_\nu(U)$. Let (e_μ) be a field of orthonormal frames on M and let (e^μ) denote the dual field of coframes. Since, by definition, $\omega_{\mu\nu}+\omega_{\nu\mu} = 0$, the 1-forms $(\omega_{\mu\nu})$ define a *metric* linear connection. Its *torsion* $\mathrm{d}e^\mu + \omega^\mu{}_\nu \wedge e^\nu$ need not be zero.

The action of the Dirac operator $\mathcal{D}$ on a spinor field is $\mathcal{D}\psi = \gamma^\mu e_\mu \lrcorner D\psi$, where $\lrcorner$ denotes contraction.

6.2. Charge conjugation and the Dirac equation

The notion of charge conjugation, defined originally by physicists for spinors associated with Minkowski space, admits a generalization to higher dimensions [E]. It is presented here for the case of an even-dimensional, flat space-time $\mathbb{R}^{2n}$ with a Lorentzian metric of signature $(2n-1,1)$. Recall (see §4.3.2) the definition of the charge conjugate φ_c of $\varphi \in S$. If φ is a Weyl spinor, then φ_c is also such a spinor and its chirality is the same as (resp., opposite to) that of φ if $\eta^2 = 1$ (resp., if $\eta^2 = -1$). If $\bar{C}C = \mathrm{id}_S$, then the map $\varphi \mapsto \varphi_c$ is involutive, $(\varphi_c)_c = \varphi$, and there is the real vector space

$$S_\mathbb{R} = \{\varphi \in S : \varphi_c = \varphi\}$$

of Dirac-Majorana spinors.

The intertwiner C satisfies

$$\bar{C}C = (-1)^{\frac{1}{2}(n-1)(n-2)}\mathrm{id}_S. \tag{24}$$

The Dirac equation for a particle of mass m and electric charge e can be written as

$$\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\psi = m\psi, \tag{25}$$

where $\psi : \mathbb{R}^{2n} \to S$ is the wave function of the particle and A_μ, $\mu = 1,\ldots,2n$, are the (real) components of the vector potential of the electromagnetic field. For a free particle ($A_\mu = 0$) one can consider a solution of (25) equal to a constant spinor times $\exp \mathrm{i}p_\mu x^\mu$; the Dirac equation then implies that the momentum vector (p_μ) is time-like: $p_{2n}^2 = p_1^2 + \cdots + p_{2n-1}^2 + m^2$. The *charge conjugate wave function* $\psi_c : \mathbb{R}^{2n} \to S$ is defined by $\psi_c(x) = \psi(x)_c$ for every $x \in \mathbb{R}^{2n}$.

Proposition 2. *If $\psi : \mathbb{R}^{2n} \to S$ is a wave function, then*
(i) *the vector field of current defined by*

$$j^\mu(\psi) = \mathrm{i}^{n+1}\langle B\gamma_{2n+1}\psi_c, \gamma^\mu\psi\rangle, \quad \mu = 1,\ldots,2n, \tag{26}$$

is real *and invariant with respect to the replacement of ψ by ψ_c,*

$$j^\mu(\psi_c) = j^\mu(\psi); \tag{27}$$

(ii) *if ψ is a solution of the Dirac equation* (25)*, then the current is conserved,*

$$\partial_\mu j^\mu(\psi) = 0, \tag{28}$$

and the charge conjugate wave function satisfies the Dirac equation for a particle of charge $-e$,

$$\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu)\psi_c = m\psi_c. \tag{29}$$

The proof of part (i) the Proposition consists of simple, algebraic transformations, making use of equations (7), (9), (10), and (24). Complex conjugating both sides of (25), multiplying the resulting equation by C^{-1} on the left and using (9) and (10) one obtains that ψ_c satisfies (29); it is then easy to check that (28) holds.

These simple observations are valid irrespective of whether the algebra $\mathrm{Cl}_{2n-1,1}$ is real ($\bar{C}C = \mathrm{id}_S$; $n \equiv 1$ or $2 \bmod 4$) or quaternionic ($\bar{C}C = -\mathrm{id}_S$; $n \equiv 0$ or $3 \bmod 4$). Charge conjugation is not related to the existence of Majorana spinors: even if the algebra $\mathrm{Cl}_{2n-1,1}$ is *real*, one has to use *complex* spinors to write the Dirac equation for a charged particle interacting with an electromagnetic field. The invariance of the current under the replacement of ψ by ψ_c, expressed by (27), reflects the classical (or rather: first-quantized) nature of the Dirac equation under consideration here. Upon second quantization, the wave function is replaced by an *anticommuting*, spinor-valued field Ψ; anticommutativity of Ψ and Ψ_c provides a change of sign, so that (27) is replaced by $j^\mu(\Psi_c) = -j^\mu(\Psi)$.

6.3. The spinorial form of the Weierstrass solution

In view of its importance, I recall here the Weierstrass solution of the equation of minimal surfaces in $\mathbb{R}^3$; further details can be found in Ch. 3 §2 of [10]. Consider a surface in $\mathbb{R}^3$ described locally by the map $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$. If u and v are coordinates on $\mathbb{R}^2$, then the line element induced on the surface is $\mathbf{r}_u \cdot \mathbf{r}_v \, \mathrm{d}u\mathrm{d}v$, where $\mathbf{r}_u = \partial\mathbf{r}/\partial u$ and $\mathbf{r}_v = \partial\mathbf{r}/\partial v$. The determinant of the metric tensor is $(\mathbf{r}_u \times \mathbf{r}_v)^2$. A *minimal surface* minimizes the area integral

$$\textstyle\int L \,\mathrm{d}u\mathrm{d}v, \text{ where } L = \sqrt{(\mathbf{r}_u \times \mathbf{r}_v)^2}.$$

The Euler–Lagrange equations for the area integral are

$$\frac{\partial}{\partial u}\frac{\partial L}{\partial \mathbf{r}_u} + \frac{\partial}{\partial v}\frac{\partial L}{\partial \mathbf{r}_v} = 0.$$

Since every surface in $\mathbb{R}^3$ is locally conformal to the plane, one can restrict the coordinates u, v so that

$$\mathbf{r}_u^2 = \mathbf{r}_v^2 \neq 0 \quad \text{and} \quad \mathbf{r}_u \cdot \mathbf{r}_v = 0. \tag{30}$$

They are defined up to transformations $u + \mathrm{i}v \mapsto f(u + \mathrm{i}v)$, where the function f is holomorphic. In these coordinates, the Euler–Lagrange equations reduce to

$$\Delta\mathbf{r} = 0, \quad \text{where} \quad \Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2.$$

Let $w = u + \mathrm{i}v$; the last equation implies the existence of $\mathbf{s} : \mathbb{R}^2 \to \mathbb{R}^3$ such that the three components of $\mathbf{r} + \mathrm{i}\mathbf{s}$ are holomorphic functions of w. Put $\mathbf{F} = \mathrm{d}(\mathbf{r} + \mathrm{i}\mathbf{s})/\mathrm{d}w$; the Cauchy–Riemann equations $\mathbf{r}_u = \mathbf{s}_v$, $\mathbf{r}_v = -\mathbf{r}_u$, together with (30), imply that the complex vector $\mathbf{F}$, with holomorphic components, is null, $\mathbf{F}^2 = 0$. Given such a vector-valued function, one has the Weierstrass solution of the equation of a minimal surface,

$$\mathbf{r}(u, v) = \mathrm{Re} \int_0^{u+\mathrm{i}v} \mathbf{F}(w')\mathrm{d}w' + \mathbf{r}(0, 0). \tag{31}$$

Paolo Budinich pointed out that there is a simple, spinorial form of the null vector appearing in the Weierstrass solution (31). Namely, if $\varphi : \mathbb{C} \to \mathbb{C}^2$ is a spinor with holomorphic components and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, then one can write

$$\mathbf{F} = \langle \varphi, \mathrm{i}\sigma_y \boldsymbol{\sigma} \varphi \rangle.$$

This equation may be obtained from Theorem 7 for $n = 1$. The papers by Budinich [3] and by Budinich and Rigoli [4] contain also an extension of this result to minimal surfaces in higher-dimensional Euclidean spaces and to strings Lorentzian spaces.

7. SPINORS IN GENERAL RELATIVITY

7.1. Almost Hermite and almost Robinson structures

Definition 4. An *N-structure* on a Riemannian manifold (M, g) of even dimension $\geqslant 4$, is a complex vector subbundle N of the complexified tangent bundle $\mathbb{C} \otimes TM$ such that, for every $p \in M$, the fiber N_p is *mtn*.

It is known that, if (M, g) is proper Riemannian, then an N-structure on M is equivalent to that of an *almost Hermite* manifold; the orthogonal almost complex structure J on M is defined as in (5) (see, e.g., Ch. IX §4 in [16]).

Definition 5. An *almost Robinson* manifold is a Lorentzian manifold with an N-structure.

In this case, the intersection $N \cap \bar{N}$ is the complexification of a line bundle $K \subset TM$; its fibers are null; they are tangent to a foliation of M by null curves. An almost Robinson structure on M is said to be *regular* if the set $\mathcal{M}$ of the leaves of the foliation defined by K has the structure of a manifold such that the natural map $\pi : M \to \mathcal{M}$ is a submersion. From now on, only such regular structures will be considered.

7.2. The integrability condition

Definition 6. The N-structure $N \to M$ on a Riemannian manifold (M, g) is said to be *integrable* if

$$[\mathrm{Sec}N, \mathrm{Sec}N] \subset \mathrm{Sec}N. \tag{32}$$

In the proper Riemannian case, condition (32) is equivalent to the vanishing of the Nijenhuis (torsion) tensor of the almost complex structure J and, by the celebrated Newlander–Nirenberg theorem, it implies that M is a Hermite manifold; see Ch. IX §2 and 4 in [16].

Definition 7. A *Robinson manifold* is an almost Robinson manifold with an integrable N-structure.

7.3. Four-dimensional Robinson manifolds: space-times with a non-distorting foliation by null geodesics

The case of dimension 4 is well known, but, since it is also the most important one, it is worth-while to review it briefly here. In this case, unlike as in higher dimensions, all information about the Robinson structure is encoded in the properties of the bundle K. I denote by k a nowhere zero section of $K \to M$: this is a null vector field. The associated 1-form is $\kappa = g(k)$.

Let (M, g) be a space- and time-oriented Robinson manifold of dimension 4 with the bundle $N \to M$ of *mtn* spaces. The fibers of the bundle $K^\perp/K \to M$ are two-dimensional 'screen spaces'. Each screen space has a complex structure, which, *in this case*, is equivalent to a conformal structure and an orientation; this being preserved by the flow is equivalent to

$$L(k)g = \rho g + \kappa \otimes \xi + \xi \otimes \kappa \tag{33}$$

for some function ρ and 1-form ξ. Physicists say that k generates a shear-free congruence of null geodesics. The expression 'shear-free' reflects the non-distorting property property of the flow: it preserves the conformal structure of the screen spaces.

'Twisting' congruences, characterized by $\mathrm{d}\kappa \wedge \kappa \neq 0$, are more interesting; the Kerr space-time, describing a black hole arising from the collapse of a rotating star, is a Robinson manifold with a twisting congruence.

Example. In Minkowski space-time, one of the first twisting shear-free congruences of null lines was described by Robinson around 1963; it played a major role in the emergence of Penrose's twistors [22,23]. Robinson established that the metric tensor

$$g = (\mathrm{d}u + \mathrm{i}(z\mathrm{d}\bar{z} - \bar{z}\mathrm{d}z))\mathrm{d}v + (v^2 + 1)\mathrm{d}z\mathrm{d}\bar{z}, \quad z = x + \mathrm{i}y \tag{34}$$

is flat and the *sng* congruence generated by ∂_v is twisting. The complex 2-form $F = A(x, y, u, v)\kappa \wedge (\mathrm{d}x + \mathrm{i}\mathrm{d}y)$ is self-dual and Maxwell's equations $\mathrm{d}F = 0$ reduce to $\partial A/\partial v = 0$ and the equation $Z \lrcorner \mathrm{d}A = 0$, where $Z = \partial_x + \mathrm{i}\partial_y - \mathrm{i}(x + \mathrm{i}y)\partial_u$ is an operator on $\mathbb{R}^3$ introduced by Hans Lewy in 1957. He constructed a smooth function h such that the equation $Z \lrcorner \mathrm{d}A = h$ has no solution, even locally.

Several solutions of Einstein's equations admit this congruence: the Gödel universe, the Taub–NUT solution and Hauser's gravitational waves of type N [17].

7.4. The Goldberg–Sachs theorem

Consider a 4-manifold (M, g) that is either proper Riemannian or Lorentzian. An N-structure on M can be (locally) given by a field φ of chiral spinors: one uses 'point by point' the definition (14).

Theorem 9. (i) *If the N-structure $N(\varphi)$ is integrable, then the chiral spinor φ is an eigenspinor of the Weyl tensor.*
(ii) *If (M, g) is conformal to an Einstein manifold, then $N(\varphi)$ is integrable if, and only if, the chiral spinor field φ is a repeated eigenspinor of the Weyl tensor.*

For space-times, the theorem was established by Goldberg and Sachs [14]. Its extension to the proper Riemannian case is due to Plebański, Hacyan, Przanowski and Broda [24,26].

8. INSTEAD OF CONCLUDING REMARKS: WORDS OF THE MASTERS

The relativity theory is based on nothing but the idea of invariance and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net, so that physical quantities exist, which it would be, to say the least, very artificial and inconvenient to express as tensors.

C. G. Darwin, The wave equation of the electron
Proc. Roy. Soc. London, **A 118** (1928) 654–680

It is interesting to note that the idea of a spinor can be based on that of a vector, and conversely that the notion of a vector can be deduced from that of a spinor; at least we can form from a pure spinor a null ν-vector (=self-dual null multivector of the middle dimension AT), then a general ν-vector can be defined as the sum of null ν-vectors, and a vector as a common element of a family of ν-vectors which satisfy certain conditions.

É. Cartan, *The Theory of Spinors*
Dover Publ., transl. by R. Streater
from the 1937 French edition

The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures.

H. Weyl, *The Classical Groups*, Princeton U.P., 1946

Spinor calculus may be regarded as applying at a deeper level of structure of space-time than that described by the standard world-tensor calculus. By comparison, world-tensors are less refined, fail to make transparent some of the subtler properties of space-time brought particularly to light by quantum mechanics and, not least, make certain types of mathematical calculations inordinately heavy.

...

Additionally, spinors seem to have profound links with complex numbers that appear in quantum mechanics.

R. Penrose, *Spinors and space-time* vol. 1
Cambridge U. P. 1984

BOOKS

[A] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, Teubner, Leipzig, 1981.

[B] H. Baum, Th. Friedrich, R. Grunewald and I. Kath, *Twistors and Killing spinors on Riemannian manifolds*, Teubner, Stuttgart–Leipzig, 1991.

[C] I. M. Benn and R. W. Tucker, *An introduction to spinors and geometry with applications to physics*, Adam Hilger, Bristol, 1988.

[D] J. D. Bjorken and S. D. Drell, *Relativistic quantum mechanics*, McGraw-Hill, New York, 1964.

[E] P. Budinich and A. Trautman, *The spinorial chessboard*, Trieste Notes in Physics, Springer, Berlin, 1988.

[F] E. Cartan, *The Theory of Spinors*, Dover Publ., New York, transl. by R. Streater from the 1937 French edition.

[G] C. C. Chevalley, *The algebraic theory of spinors*, Columbia Univ. Press, Morningside Heights, NY, 1954.

[H] L. Dąbrowski, *Group actions on spinors*, Bibliopolis, Napoli, 1988.

[I] Th. Friedrich, *Dirac operators in Riemannian geometry*, Grad. Studies in Math. 25, Amer. Math. Soc., Providence, RI, 2000.

[J] H. B. Lawson, Jr and M.-L. Michelsohn, *Spin geometry*, Princeton Univ. Press, Princeton, NJ, 1989.

[K] R. Penrose and W. Rindler, *Spinors and space-time*, vols 1 and 2, Cambridge Univ. Press, Cambridge, 1984 and 1986.

[L] S. Weinberg, *The quantum theory of fields*, vol. 3, Cambridge Univ. Press, Cambridge

[M] H. Weyl, *The classical groups*, Princeton Univ. Press, Princeton, NJ, 1946.

REFERENCES

[1] J. F. Adams, *Spin(8), triality, F_4 and all that*, Superspace and Supergravity (S. W. Hawking and M. Roček, eds.), Cambridge University Press, 1981.

[2] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), no. Suppl. 1, 3–38.

[3] P. Budinich, *Null vectors, spinors, and strings*, Commun. Math. Phys. **107** (1986), 455–465.

[4] P. Budinich and M. Rigoli, *Cartan spinors, minimal surfaces and strings*, Nuovo Cimento **102 B** (1988), 609–647.

[5] P. Budinich and A. Trautman, *Remarks on pure spinors*, Lett. Math. Phys. **11** (1986), 315–324.

[6] ______, *An introduction to the spinorial chessboard*, J. Geom. Phys. **4** (1987), 361–390.

[7] ______, *Fock space description of simple spinors*, J. Math. Phys. **30** (1989), 2125–2131.

[8] É. Cartan, *Sur les groupe projectifs qui ne laissent invariante aucune multiplicité plane*, Bull. Soc. Math. **41** (1913), 53–96.

[9] ______, *Sur les espaces conformes généralisés et l'univers optique*, C. R. Acad. Sci. Paris **174** (1922), 857–859.

[10] R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. 2, Springer, Berlin, 1937.

[11] P. A. M. Dirac, *The quantum theory of the electron*, Proc. Roy. Soc. London A **117** (1928), 610–624.

[12] T. Dray and C. A. Monague, *Octonionic Möbius transformations*, Modern Phys. Lett. A **14** (1999), 1243–1255.

[13] Th. Friedrich and A. Trautman, *Spin spaces, Lipschitz groups, and spinor bundles*, Ann. Glob. Anal. Geom. **18** (2000), 221–240.

[14] J. N. Goldberg and R. K. Sachs, *A theorem on Petrov types*, Acta Phys. Polon. A Suppl. **22** (1962), 13–23.

[15] O. Hijazi, *Lower bounds for the eigenvalues of the Dirac operator*, J. Geom. Phys. **16** (1995), 27–38.

[16] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. I and II, Interscience, New York, 1963 and 1969.

[17] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact solutions of Einstein's equations*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980.

[18] R. O. Lipschitz, *Untersuchungen über die Summen von Quadraten*, Max Cohen und Sohn, Berlin, 1886, French summary of the book in: Bull. Sci. Math. **21** (1886) 163–183.

[19] W. Pauli, *Zur Quantenmechanik des magnetischen Elektrons*, Z. Physik **43** (1927), 601–623.
[20] ———, *Contributions mathématiques à la théorie des matrices de Dirac*, Ann. Inst. H. Poincaré **6** (1936), 109–136.
[21] R. Penrose, *A spinor approach to general relativity*, Ann. Physics **10** (1960), 171–201.
[22] ———, *Twistor algebra*, J. Math. Phys. **8** (1967), 345–366.
[23] ———, *On the origins of twistor theory*, Gravitation and Geometry (W. Rindler and A. Trautman, eds.), Bibliopolis, Napoli, 1987, A volume in honour of Ivor Robinson, pp. 341–361.
[24] J. F. Plebański and S. Hacyan, *Null geodesic surfaces and the Goldberg–Sachs theorem in complex Riemannian spaces*, J. Math. Phys. **16** (1975), 2403–2407.
[25] I. R. Porteous, *Topological geometry*, Van Nostrand, London, 1969.
[26] M. Przanowski and B. Broda, *Locally Kähler gravitational instantons*, Acta Phys. Polon. **B 14** (1983), 637–.
[27] M. Riesz, *L'équation de Dirac en relativité générale*, C. R. XII Congrès des Mathématiciens Scandinaves (Lund), 1953, pp. 241–259.
[28] I. Robinson and A. Trautman, *The conformal geometry of complex quadrics and the fractional-linear form of Möbius transformations*, J. Math. Phys. **34** (1993), 5391–5406.
[29] F. Sauter, *Lösung der Diracschen Gleichungen ohne Spezialisierung der Diracschen Operatoren*, Z. Phys. **63** (1930), 803–814.
[30] J. Tits, *Sur la trialité et certains groupes qui s'en déduisent*, Publ. Math. IHES (1959), no. 2, 13–60.
[31] A. Trautman, *Spinors and the Dirac operator on hypersurfaces. I. General theory*, J. Math. Phys. **33** (1992), 4011–4019.
[32] ———, *Clifford and the 'square root' ideas*, Contemporary Mathematics **203** (1997), 3–24.
[33] ———, *Pythagorean spinors and Penrose twistors*, The Geometric Universe: Science, Geometry and the Work of Roger Penrose (S.A. Huggett, L.J. Mason, K.P. Tod, S.T. Tsou, and N.M.J. Woodhouse, eds.), Oxford Univ. Press, Oxford, 1998, pp. 411–419.
[34] ———, *Reflections and spinors on manifolds*, Proc. of conference CP453: Particles, Fields, and Gravitation (J. Rembieliński, ed.), The American Institute of Physics, 1998, pp. 518–527.
[35] A. Trautman and K. Trautman, *Generalized pure spinors*, J. Geom. Phys. **15** (1994), 1–22.
[36] B. L. van der Waerden, *Spinoranalyse*, Nach. Ges. Wiss. Götingen, math. phys. Kl. (1929), 100.
[37] ———, *Exclusion principle and spin*, Theoretical Physics in the Twentieth Century: A Memorial Volume to Wolfgang Pauli (M. Fierz and V.F. Weisskopf, eds.), Interscience, New York, 1960, pp. 199–244.
[38] O. Veblen and J. W. Givens, *Geometry of complex domains*, notes from a seminar conducted in 1935-36, Institute for Advanced Study, Princeton, NJ, 1955.
[39] C. T. C. Wall, *Graded Brauer groups*, J. reine und angew. Math. **213** (1964), 187–199.
[40] A. Weil, *Correspondence, from an ultramundane correspondent*, Œuvres Scientifiques (Collected Papers), Vol. II (1951-64), Springer, 1979, reprinted from *Ann. of Math.* **69** (1959) 247–251.