Andrzej.Trautman@fuw.edu.pl
Institute of Theoretical Physics, Warsaw University
Hoża 69, PL-00681 Warszawa

ELEMENTARY APPROACH TO THE IDEA OF GENERAL RELATIVITY

Full GRT requires Riemannian geometry

Einstein's eqs for the gravit. field are a hard-to-solve system of nonlinear PDEs for 10 functions $g_{\mu\nu}$ of 4 space-time coordinates

But:

- there are vestiges of general relativity in Newtonian gravitation
- application to Newtonian cosmology (expansion and big bang)
- propagation of light in a gravit. field forces curvature of space-time
- motion with constant acceleration in SRT leads to horizon (black holes in GRT)
- weakness of gravit. radiation and "large numbers"

3 kinds of masses in Newton's equations:

$$m_{\rm in}\ddot{\mathbf{r}} = \mathbf{F} \qquad \text{in general}$$

$$= m_{\rm p.gr}\mathbf{g} \qquad \text{in a grav. field}$$

$$\mathbf{g} = -\frac{Gm_{\rm a.gr}}{r^2}\frac{\mathbf{r}}{r} \qquad \text{Newton's law}$$

$m_{\rm p.gr} = m_{\rm a.gr}$ follows from equality of action and reaction (conservation of momentum), but

$m_{\rm in} = m_{\rm gr}$ follows from experiments (Galileo and the leaning tower of Pisa (?), Eötvös, Dicke, Braginsky,. . .); equality established with accuracy $\sim 10^{-12}$; contrast with electricity: $m\ddot{\mathbf{r}} = q\mathbf{E}$.

Nature is telling us something: Einstein understood what it was.

From now on $m = m_{\rm in} = m_{\rm gr}$. Mass drops out of equation of free falls:

$$\ddot{\mathbf{r}} = -\mathrm{grad}\,\varphi, \quad \varphi = \text{grav. potential}$$

This equation is invariant with respect to (time-independent) rotations in space and also with respect to

$$\mathbf{r} \mapsto \mathbf{r}' = \mathbf{r} - \mathbf{a}(t), \quad \varphi \mapsto \varphi' = \varphi + \ddot{\mathbf{a}} \cdot \mathbf{r} + \mathbf{b}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary functions of time.

The arbitrariness of $\mathbf{a}$ is the Newtonian residue of Einstein's general covariance.

Consequence: since

$$\operatorname{grad}\varphi \mapsto \operatorname{grad}\varphi' = \operatorname{grad}\varphi + \ddot{\mathbf{a}}\ .$$

the gravitational force $\operatorname{grad}\varphi$ can be locally eliminated by choice of reference frame (Einstein's falling elevator; weightlessness in artificial satellites, an experience everyone can now buy for 20 million dollars).

The differential gravitational forces, responsible for tides, are invariant:

$$\frac{\partial^2\varphi'}{\partial x_i \partial x_k} = \frac{\partial^2\varphi}{\partial x_i \partial x_k}, \qquad (x_1, x_2, x_3) = \mathbf{r}.$$

The correspondence between the Newton and Einstein theories is roughly as follows

φ corresponds to a part of $g_{\mu\nu}$

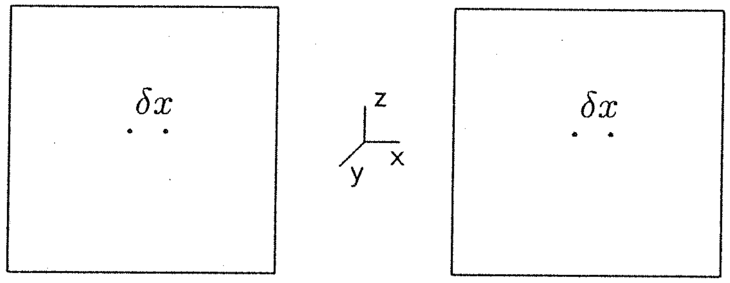
$\operatorname{grad}\varphi$ to Christoffel symbols and

$\partial^2\varphi/\partial x_i \partial x_k$ to Riemann tensor.

How to predict one's fate in a weightless situation?

Physics in a freely falling windowless laboratory

free motion in empty space | free fall towards a star



$\delta x, \delta y, \delta z =$ const.

$\delta x, \delta y \searrow \ \delta z \nearrow$

One has $\delta\ddot{x} = -\frac{\partial^2\varphi}{\partial x^2}\delta x$, etc.

Experiment gives

$$\delta\ddot{x} + \delta\ddot{y} + \delta\ddot{y} = 0$$

this implies the equation of Newtonian gravitation in empty space:

$$\Delta\varphi \stackrel{\text{def}}{=} \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0.$$

Newtonian gravitation locally described by

$$\ddot{\mathbf{r}} = -\mathrm{grad}\,\varphi, \quad \Delta\varphi = 4\pi \mathrm{G}\rho, \quad \rho = \text{density of mass}$$

defined up to transformations

(GG) $\mathbf{r}' = \mathbf{r} - \mathbf{a}(t), \quad \varphi' = \varphi + \ddot{\mathbf{a}} \cdot \mathbf{r} + \mathbf{b}, \quad \rho' = \rho$

For an isolated system of masses one can restrict φ by imposing the condition

(B) $\lim_{r\to\infty} \varphi = 0$ so that $\ddot{\mathbf{a}} = 0$ and $\mathbf{b} = 0$.

This determines $\mathbf{r}$ up to Galileo transformations,

$$\ddot{\mathbf{a}} = 0 \quad \text{and} \quad \mathbf{b} = 0 \quad \text{implies} \quad \mathbf{r}' = \mathbf{r} - \mathbf{V}t - \mathbf{r}_0$$

But in cosmology, where one considers the Universe filled uniformly with matter, one cannot assume (B). Failure to understand this prevented the development of Newtonian cosmology.

Cosmology is based on the cosmological principle: an assumption of homogeneity and isotropy of the average distribution of matter and of its motion. It implies $\rho = \rho(t)$; a simple solution of $\Delta\varphi = 4\pi G\rho$ is

(C1) $\varphi = \frac{2}{3}\pi G\rho r^2.$

Assume also that the velocity of the cosmological substratum is

(C2) $\mathbf{v}(\mathbf{r}, t) = \mathbf{r}\dot{R}(t)/R(t)$ Hubble's law

At first sight assumptions (C1) and (C2) seem to violate the cosmological principle. But they do not by virtue of the freedom in changing reference frames larger than implied by Galileo transformations. Namely, fix any time t_1, position $\mathbf{r}_1$ and assume $R(t_1) = 1$. The solution of $d\mathbf{r}/dt = \mathbf{v}$ satisfying $\mathbf{r}(t_1) = \mathbf{r}_1$ with $\mathbf{v}$ given by (C2) is

$$\mathbf{r}(t) = R(t)\mathbf{r}_1.$$

Using this to perform a generalized Galileo transformation (GG), $\mathbf{r}' = \mathbf{r} - \mathbf{a}$ with $\mathbf{a}(t) = R(t)\mathbf{r}_1$, one obtains

(C2') $\mathbf{v}'(\mathbf{r}', t) = \mathbf{r}'\dot{R}(t)/R(t)$

so that every observer moving with the substratum sees the same velocity distribution. One also checks that φ' and φ are connected by (GG).

The conservation law of energy for a unit mass at distance $R(t)$ is

$$(\mathrm{F}) \qquad \frac{1}{2}\dot{R}^2 - \frac{GM}{R} = E = \text{const.}$$

where

$$M = \frac{4\pi}{3}\rho R^3 = \text{const.}$$

is the mass inside a sphere of radius R. Equation (F) is identical in form to the one derived by Friedmann in full GRT; the only additional information provided by Einstein's equations is that $-E \sim$ curvature of space ($E < 0$ corresponds to a closed Universe). In the Newtonian case it is natural to take $E = 0$ and then $R(t) \sim t^{2/3}$: there is expansion from a singularity (big bang) at $t = 0$. For $E < 0$ expansion is followed by contraction ('big crunch').

There is a quasi-derivation of the general-relativistic Friedmann line-element from the Newtonian velocity field (C2); namely,

$$dt^2 - (d\mathbf{r} - \mathbf{v}dt)^2 = dt^2 - R(t)^2 d\mathbf{r}'^2,$$

where $\mathbf{r}' = \mathbf{r}/R$ is a coordinate system comoving with the substratum.

Relativistic gravitation requires curved space-time

Quantum theory tells us that light – and all other physical phenomena – have two aspects: corpuscular and wave.

Photons – particles of light – corresponding to an electromagnetic wave of frequency ν have energy $h\nu$.

Recall conservation of energy in a static grav. field:

$$m(\ddot{\mathbf{r}} + \mathrm{grad}\,\varphi) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{dt}}(\tfrac{1}{2}\mathrm{m}\dot{\mathbf{r}}^2 + \mathrm{m}\varphi) = 0.$$

Creation and annihilation of pairs:

electron+positron$\leftrightarrow$ two photons

Consider the following cycle of events in the (approximately) static grav. field near the surface of the Earth:

at potential $\varphi + \delta\varphi$ there is at rest an electron and a positron, each of mass m.

the pair falls down to where the potential is φ; each of the particles acquires a kinetic energy $m\delta\varphi$.

the pair annihilates at the lower level to produce two photons of frequency $\bar{\nu}$ so that $h\bar{\nu} = mc^2 + m\delta\varphi$.

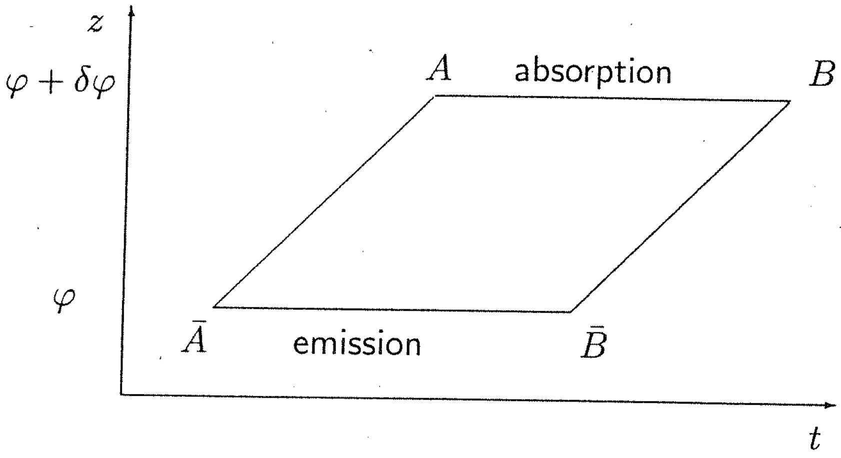
the photons are now directed upwards to where the potential is $\varphi + \delta\varphi$; their frequency diminishes to $\nu = mc^2/h$. (Otherwise there would be a *perpetuum mobile*!).

Conclusion: photons moving up in the grav. field undergo a red shift,

$$\bar{\nu} - \nu = \nu \delta\varphi / c^2.$$

This has been observed, first in the light emitted near the surface of stars and, in the 1960s, in laboratory experiments using the Mössbauer effect (Pound and Rebka).

Consider now the same phenomenon from the dual point of view of the wave theory of light. Let z be a vertical axis and represent events in the t, z plane.



Assume there is a monochromatic wave of frequency $\bar{\nu}$ emitted upwards; then $\bar{A}\bar{B} \| AB$ because emitter and absorber at rest and $\bar{A}A \| \bar{B}B$ because gravit. field is stationary; if $\bar{A}\bar{B} = 1/\bar{\nu}$, then $AB = 1/\nu$ so that $AB \neq \bar{A}\bar{B}$ and space-time is curved

Horizons and black holes

Laplace (1795) noticed that the escape velocity v of a particle from the surface of a body of mass m and radius r is determined by

$$\frac{1}{2}v^2 - \frac{Gm}{r} = 0.$$

Therefore, if

$$r < r_{\rm grav} = \frac{2Gm}{c^2}, \quad \text{then} \quad v > c$$

and light cannot escape from the surface of a body of radius $r < r_{\rm grav}$.

Look at uniformly accelerated motion in special relativity: its world-line is one branch of the hyperbola

$$\text{(H)} \quad c^2t^2 = x^2 - l^2, \quad x > 0 \quad \text{so that} \quad x \approx l + \tfrac{1}{2}at^2,$$

where $a = c^2/l$ is the acceleration. The asymptote of equation $ct = x$ is an event horizon: light from events with $ct > x$ cannot reach an observer moving along (H), but the observer can send signals to that region of space-time. There is also a particle horizon given by the other asymptote of equation $ct = -x$: the observer receives signals from the region $ct < -x$, but cannot send any particle to it.

The equality $m_{\text{in}} = m_{gr}$ implies that, locally, inertial and gravitational forces cannot be distinguished (principle of equivalence).

Therefore, a horizon can occur in a gravitational field without invoking uniform acceleration: an observer at rest near the surface of a body feels a constant pull, equivalent to a constant acceleration. But the Laplace argument shows that for the horizon to appear, the body should have a radius smaller than r_{gr}. For a long time this possibility was dismissed as never being realized in Nature (for the Sun, r_{gr} is about 1.5 km and 10^{-54} cm for a neutron).

In 1931-32 S. Chandrasekhar and L. D. Landau have shown that 'old stars' that have burned out its nuclear fuel can remain in equilibrium, due to the pressure of their constituent degenerate gas of fermions (Pauli exclusion principle!) provided that their masses do no exceed a maximal mass of the order of

$$M_{\text{max}} = m\left(\frac{Gm^2}{\hbar c}\right)^{-3/2}, \quad \text{where } m \text{ is the nucleon mass.}$$

White dwarfs and neutron stars.

Note the appearance of the gravitational fine structure constant

$$\frac{Gm^2}{\hbar c} \approx 10^{-40}$$

an analog of the (electromagnetic) fine structure constant $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$.

1939: Oppenheimer and Snyder showed that a ball of dust collapses under the influence of gravitation and 'disappears' inside its horizon. (Result ignored by Einstein? Perfect dust unphysical) A singularity is formed; what is its meaning? Classical theory fails; quantum evaporation of black holes (Hawking radiation); cosmic censorship.

Major advances in astrophysics starting In the 1960s

- 1965: discovery of relict radiation – support for big bang cosmology
- 1968: pulsars discovered and recognized as neutron stars
- quasars and some galactic centers are sources of enormous energy
- black holes
  - are formed in collapse of old stars with mass larger than M_{max}
  - are sources of energy of quasars?
  - are observed as black companions of double stars (first: Cygnus X1)
  - of very large mass are in centers of many galaxies

Gravitational radiation

What is the source of the gravitational field in relativistic situations? In Newton's theory it is the density of mass; in SRT $mc^2 = E$ and energy forms a vector with momentum: the total energy-momentum vector of an isolated system is conserved. In electrodynamics, charge is the basic conserved quantity and charges, with the associated currents, are sources of the e.m. field. In GRT, its the density of energy-momentum, together with their currents, are the sources of the grav. field.

The simplest radiating system consists of two bodies of equal mass m (in electrodynamics particles of charge e and $-e$) moving on a circle of radius r around their common center of mass.

Conservation of charge excludes monopole (spherically symmetric) radiation in electrodynamics.

Conservation of energy and momentum excludes monopole and dipole radiation in grav. theory.

Emission of energy per unit of time

e.m. radiation $\sim \frac{mc^2}{r/c}\left(\frac{e^2}{mc^2 r}\right)^3$

grav. radiation $\sim \frac{mc^2}{r/c}\left(\frac{Gm}{c^2 r}\right)^4$

For an electron ratio of fine structure constants: $\frac{Gm^2}{e^2} \approx 10^{-42}$.

Large numbers, accidental coincidences and the Dirac hypothesis

Belief among physicists: pure numbers such as

$$e^2/\hbar c \approx 1/137, \quad m_{\text{proton}}/m_{\text{electron}} \approx 1836$$

should be 'explained' = calculated from first principles.

But what about very large numbers:

$$(\text{grav. fine structure const.})^{-1} = \hbar c/Gm^2_{\text{proton}} \approx 10^{40}$$

$$e^2/Gm_{\text{proton}}m_{\text{electron}} \approx 0.2\ 10^{40}$$

$$\text{“age of Universe”/atomic unit of time} \approx 10^{40}$$

Dirac: the last number is the key: in all other numbers there appears the gravitational constant that varies with time, $G(t) \sim 1/t$. But observations do not confirm the Dirac hypothesis. . .

The anthropic principle: if the fundamental constants of physics had been different from what they are, intelligent life could not have developed and there would be no one to ask this question. An explanation without explanation.

A major theoretical issue: develop a quantum theory of space-time and gravitation. Supergravity? Superstrings? Non-commutative geometry?