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Gravitational Waves and Radiation

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Introduction

These notes contain a review of some recent papers on the theory of gravitational waves and radiation. No attempt has been made to describe here any of the proposals and endeavours to generate and detect gravitational waves. The interested reader should consult the works of Weber (1961) and Braginski (Mirianashvili, 1965).

It was a rather hard task to decide which papers should be reviewed. A well-balanced survey of all research done on gravitational waves since 1916 was beyond the author's possibilities. On the other hand, it seemed unwise to restrict oneself only to what has been done since the last conference. The most recent papers, important as they are, appear to be too specialized to form by themselves the core of a general survey. As a compromise, we decided to present a selection of classical results on waves, with emphasis on those that have a bearing on recent research. Many important papers are not even mentioned here; in any particular case this may be due to one of the two reasons: 1. a conviction that the paper in question is a classic and its content is widely known or readily available in the existing reviews; 2. the present author's ignorance. Moreover, we consciously avoid those topics that will be covered by other reviewers at this conference: exact radiative solutions, conservation laws, and the relation between classical and quantum descriptions of radiation.

The first two sections contain some elementary arguments and estimates of the magnitude and nature of gravitational radiation. No definite theory of gravitation is assumed there. The rest of the text is devoted to waves within the framework of Einstein's theory.

1. Gravitational radiation is at least of the quadrupole type.

The origin of our ideas on gravitational waves and radiation may be traced to the similarity between electromagnetic and gravitational interactions. As emphasized by Feynman (1957), it is possible to draw a number of conclusions about the gravitational field by applying to it the conventional methods of field theory in flat space. There are indications that general relativity, in a certain sense, may be derived from field theory (Thirring, 1959; Sexl, 1961; Halpern, 1963).

The similarity between electromagnetism and gravitation is most striking for non-relativistic phenomena. Since gravitation exists as a classical (macroscopic) field, one expects it to be describable by a tensor field. From the $1/r^2$ and attractive character of the gravitational force one infers that it corresponds to a mass-zero, even spin field. A further elementary fact is the equality of gravitational and inertial masses, i.e., the experimentally established proportionality between inertial masses and gravitational "charges". This, together with the Newtonian law of conservation of mass, implies the absence of gravitational monopole and dipole radiation and substantiates the choice of a spin-two field to describe gravitation. Indeed, let ϕ be that component of the gravitational field which in the non relativistic limit goes over into the Newtonian potential. For not too large velocities and not too strong fields, one can expect ϕ to satisfy

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 4\pi k\rho, \tag{1}$$

where k is the gravitational constant and ρ the density of mass. The expansion into multipoles of a retarded solution of (1) is

$$\phi = -\frac{kM}{r} - \frac{k\,\underline{r}\cdot\underline{P}}{c r^2} \quad + \text{higher multipoles.} \tag{2}$$

Here M and $\vec{P}$ are the total mass and momentum, respectively. For an isolated system these quantities are constant. They would not have to be such if the source of the gravitational field had not been identified with the distribution of inertial mass. It follows from field theory that the flux of radiated energy is given by an expression quadratic in the first derivatives of the potentials. Accordingly, one can expect the intensity P of radiation by gravitational waves to be of the order of

$$\frac{c}{k} \oint (\nabla \phi)^2 \, dS, \tag{3}$$

where the coefficient c/k has been introduced to ensure the correct dimension; the integral is extended over the surface of a large sphere surrounding the system. The monopole and dipole terms, as given by (2), have derivatives behaving like $1/r^2$ and do not contribute to (3). Gravitational radiation is predominantly of the quadrupole type; if D_{ij} $(i,j = 1,2,3)$ is the tensor of quadrupole moment, then

$$P \sim \frac{k}{c^5} \dddot{D}_{ij} \dddot{D}_{ij} \tag{4}$$

(plus contributions from higher multipoles). Formula (4) is a simple consequence of (2) and (3) and does not depend on any particular theory of gravitation (Weber, 1961). Such a theory is necessary to fix the numerical coefficient in (4). It is sometimes argued that the absence of gravitational dipole radiation is due to the non-existence of negative masses. The above argument shows that this is not so and that the equality of gravitational and inertial masses is essential. There would be no dipole radiation, even with negative masses, provided this equality held. It is also possible to give an argument to support the converse: the sources of a spin-two, mass-zero field must be identified with the distribution of energy and momentum. When applied to point particles this gives the equality of gravitational mass and inertial mass (Weinberg 1964).

2. Smallness of gravitational radii.

The amount of gravitational radiation predicted by theory is known to be notoriously small. One reason why this is so has been given in the preceding section. A more important reason is the smallness of gravitational radii, km/c^2.

For a gravitating system consisting of two bodies of equal mass m moving about one another along circular orbits of radius r, formula (4) gives

$$P \simeq \frac{mc^2}{r/c}\left(\frac{km}{c^2 r}\right)^4.$$

For a similar system in electrodynamics (two particles of equal mass m and opposite charges e and -e, moving in circular orbits under their own attraction, with a velocity $v \ll c$), the amount of electromagnetic (dipole) radiation is

$$P_{em} \simeq \frac{mc^2}{r/c}\left(\frac{e^2}{mc^2 r}\right)^3$$

and the amount of gravitational radiation

$$P \simeq \frac{km}{c^2 r} P_{em}.$$

As another example we may quote the non-relativistic motion of a charge in an external, constant magnetic field. If v is the velocity and r the radius of the orbit, one obtains for the power of the electromagnetic and gravitational radiation

$$P_{em} \simeq \frac{mc^2}{r/c}\left(\frac{v}{c}\right)^4 \frac{e^2}{mc^2 r} \quad \& \quad P \simeq \frac{km^2}{e^2} P_{em}.$$

respectively (Postovoit and Gercenstein, 1962).

From these and similar examples one can infer that the magnitude of radiation depends in an essential way on the ratio of the gravitational (respectively, electromagnetic) radius of the source to a length characterizing the dimensions of the system.

For atomic systems

$$\frac{e^2}{mc^2 r} \sim \alpha^2 \sim 10^{-4},$$

$$\frac{km}{c^2 r} \sim 10^{-47}.$$

(e and m refer to the electron and r is the radius of an atom).

A double-star system may radiate a very significant amount of gravitational energy provided that its components are super dense ($km/c^2 r \sim 1$ on the surface of the star) and move grazing one another.

Approximative methods of finding gravitational waves

Assuming a definite theory of gravitation, it is possible gravitational waves from a bounded material system. This is quite easy in the linear theory of a mass-zero, spin-two field. One then takes the canonical energy-momentum tensor to determine the flux of energy and obtains the formula

$$P = \frac{1}{45}\frac{k}{c^5}\dddot{D}_{ij}\dddot{D}_{ij} \tag{5}$$

where higher order multipoles have been neglected.

The situation is not so straightforward in general relativity where the field equations are so complicated that no exact, realistic radiative solutions could have been found. Moreover, the notion of gravitational energy in that theory is somewhat abscure. To evaluate the radiated power it is necessary to set up an approximate method of solving the field equations and to give a prescription how to compute the change in the total energy of the system. Various approaches to gravitational radiation can be classified according to the methods used to solve these two problems.

3. Weak-field solutions.

Einstein (1916, 1918) gave a prescription how to construct approximate solutions of the field equations valid for weak fields: he normalized the co-ordinates in a certain way, put $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($\mu,\nu = 0,1,2,3$) and neglected all terms in $G_{\mu\nu}$ non-linear in the h's. Taking a retarded solution of the linearized equations, introducing it into the canonical energy-momentum "pseudotensor", and integrating the resulting Poynting vector over a large sphere gives again (5). This approach has been criticized on several grounds: The weak-field approximation neglects an essential feature of Einstein's equations, their non-linearity and is really equivalent to replacing general relativity by a linear theory of gravitation in flat space. This theory allows strictly periodic radiative fields whereas it is clear that radiation in general relativity must be accompanied by secular effects. In a different context, Synge (1960) proposed the following interpretation of the approximate solutions: consider them as exact solutions for a different matter distribution, to be determined from Einstein's equations. If this is done for a radiative weak-field solution, one finds that the corresponding flux of matter exactly counterbalances the outflow of gravitational radiation, as computed from the pseudotensor.

It seems that a sound point of view is to consider the weak-field solutions as a first step of a consistent approximation scheme ("fast motion approximation"). The foundations of such a scheme have been developed by several authors (Bertotti and Plebanski, 1960, Goldberg and Havas, 1962). In the second approximation, the radiative corrections to the motion of point particles have been obtained (Havas, 1957). However, it is far from clear whether the method converges or even whether it can be continued, in the general case beyond the first step. To substantiate these doubts, one may argue as follows: a typical component of $\underset{1}{h}_{\mu\nu}$ has the form of an outgoing wave, $\underset{1}{a}(t-r)/r$; assume that $\underset{1}{a}(t)$ vanishes outside the interval (t_o, t_1). In harmonic co-ordinates, the equation for the second order correction $\underset{2}{h}_{\mu\nu}$ is, symbolically,

$$\Box \underset{2}{h} = Q(\underset{1}{h}) \tag{6}$$

where $Q(\underset{1}{h})$ is an expression quadratic in $\underset{1}{h}$ and its derivatives, and $\Box$ is the wave operator in flat space. If we put $\underset{2}{h} = \psi/r$, introduce null coordinates $u = t-r$ and $v = t+r$ and neglect in Q terms of the order $O(1/r^3)$, then (6) may be written as

$$\frac{\partial^2\psi}{\partial u\,\partial v} = \frac{f(u)}{v-u},$$

where $f \sim \dot{\underset{1}{a}}^2$. The last equation may be integrated to yield

$$\psi(u,v) = \underset{2}{a}(u) + \underset{2}{b}(v) + \int_{t_0}^{u} f(t)\log(v-t)\,dt,$$

where $\underset{2}{a}$ and $\underset{2}{b}$ are arbitrary functions. We may rule $\underset{2}{b}$ out on the ground that it corresponds to an incoming wave. For $u = \text{const.} > t_o$ and large v, $v \gg t_1 - t_o$, one has (Fock, 1955; Trautman, 1958)

$$\psi \simeq \log(v-t_0)\int_{t_0}^{u} f(t)\,dt .$$

In other words, for large r and t-r = const., the second order correction may behave like log r/r. If this behaviour were to be characteristic also of the exact metric, it would contradict the Sommerfeld radiation condition and make it impossible to compute the flux of radiation. It may very well be so that this difficulty can be resolved by choosing coordinate conditions other than the harmonic ones.

4. The Einstein-Infeld-Hoffmann approximation method.

The fast motion approximation method is not well suited for systems consisting of freely gravitating bodies such as the planetary system. In the first order, the equations of motion obtained by this method are trivial (no interaction). Einstein, Infeld and Hoffmann (1938) and Fock (1939) devised a new approximation method which gives the Newtonian equations of motion in the first step. This is achieved by considering terms such as

$$(v/c)^2 \quad \text{and} \quad km/c^2r \tag{7}$$

as being of the same (second) order of magnitude. Formally, the EIH approach consists in expanding all functions in power series in $1/c$. The EIH method is suitable (i.e. it converges fast) for situations such that both expressions (7) are small. This implies, in particular, that one should have

$$km/c^2 \ll r \ll \lambda$$

where $T = \lambda/c$ is a time interval characteristic for the system under consideration. It follows that the method is not well suited to evaluate radiation by means of a surface integral such as (3): the integral should be taken over the surface of a sphere in the wave zone, i.e., for $r \gg \lambda$. From simple heuristic arguments one infers that the first few terms of the expansion of the components of the metric tensor are

$$g_{oo} = 1 + \underset{2}{g}_{oo} + \underset{4}{g}_{oo} + \dots$$

$$g_{ok} = \underset{3}{g}_{ok} + \underset{5}{g}_{ok} + \dots$$

$$g_{ik} = -\delta_{ik} + \underset{2}{g}_{ik} + \underset{4}{g}_{ik} + \dots$$

and that the first terms which may correspond to gravitational radiation are $\underset{5}{g}_{ik}$, $\underset{6}{g}_{ok}$, $\underset{7}{g}_{oo}$ (Infeld, 1938; Hu, 1947; Goldberg 1955; Trautman 1958b). Whether these terms are genuine or trivial, i.e., reducible to zero by a coordinate transformation, depends on whether the linearized curvature tensor

$$\underset{7}{g}_{oo,ik} + \underset{5}{g}_{ik,oo} - \underset{6}{g}_{io,ko} - \underset{6}{g}_{ko,io}$$

arbitration in the choice of these radiative fields; this corresponds to the freedom in boundary conditions. Once the fields up to a certain order are given, it is possible to derive the corresponding equations of motion (see, e.g., Infeld and Plebanski, 1960). The radiative fields ($\underset{5}{g}_{ik}$, $\underset{6}{g}_{ok}$, $\underset{7}{g}_{oo}$) lead to a damping force in the 9th order, the Newtonian equations being considered as of the 4th order. A reasonable choice of the radiative terms for a two-body system gives a damping force whose magnitude is in agreement with the loss of energy for this system, as evaluated from (5). This result, due to Peres (1960), substantiates the validity of the weak-field approach to gravitational radiation.

5. A mechanical description of radiation damping

The equations of motion obtained by the EIH method are of a mechanical type: they contain quantities referring only to particles. Up to a certain order these equations may be derived from a lgrangian which is invariant under space and time translations. This implies that to that order the total energy and momentum are conserved and the system does not radiate. A similar connection between radiation and motion is familiar from electrodynamics: for a system of interacting charges there exists a mechanical lagrangian giving their motion with a "post-Coulombian" accuracy (Darwin, 1920). In general, a conservative and more accurate electromagnetic lagrangian does not exist; it can be introduced only when the interactions are assumed to be of the half-retarded, half-advanced type. (Fokker, 1929; Wheeler and Feynman, 1949). In general relativity this problem was considered by Fock (1955), Infeld (1957), Infeld and Plebanski (1960) and Plebanski and Bazanski (1959).

Recently, L. Infeld and R. Trautman (1965) have analyzed in considerable detail the connection between radiation and the possibility of introducing a generalized mechanical lagrangian for a particle. They propose a method which may be used, in conjunction with the EIH technique, to evaluate the energy and momentum radiated by a particle in the form of gravitational electromagnetic or other waves. The following lines contain a brief description of the method.

To alleviate the formulae, a symbolic notation will be used here: ψ will denote a field or a set of fields with indices suppressed; ψ' will represent the set of first partial derivatives of ψ with respect to the coordinates x^{μ}; ξ will stand for the three spatial coordinates of a particle. In many expressions a summation over the suppressed indices will be taken for granted. It will be assumed that there is given a privileged time coordinate t; e.g., it may be defined by an external static field. Let the action for the system consisting of a particle and the field be

$$W = \int_{t_1}^{t_2} dt \left(\Lambda + \int_V \mathcal{L}\, dx \right)$$

where

$$\Lambda = \Lambda(\xi, \dot{\xi}, \psi), \quad \mathcal{L} = \mathcal{L}(\psi, \psi'),$$

V is a three-dimensional region which will later be assumed to coincide with the whole space t = const.; dx is an element of volume in V and $\dot{\xi} = \frac{d\xi}{dt}$. By varying W with respect to ξ and ψ one obtains the equations of motion

(8) $\Omega = 0$, where $\Omega(\xi, \dot{\xi}, \ddot{\xi}, \psi, \psi') = \frac{\delta W}{\delta \xi}$,

and the field equations,

(9) $\Phi = 0$, where $\Phi(\xi, \dot{\xi}, \psi, \psi', \psi'') = \frac{\delta W}{\delta \psi}$.

The ψs occurring in Ω should be evaluated at the point $x = \xi$. The equations (8) and (9) are coupled and should be solved simultaneously; in fact, in general relativity they cannot be solved otherwise. In special relativity, and also in general relativity when an appropriate approximation scheme is employed, it is possible to find a solution of the field equation $\Phi = 0$ corresponding to an arbitrary motion $\xi(t)$.

Let

(1) $$\psi = \Psi(x, \xi, \dot{\xi})$$

be such a solution. Following Infeld, we assume here that Ψ does not depend explicitly on t and is a rather simple function of the motion of the particle. These assumptions are fulfilled within the framework of the ETH method. Upon substituting (10) into (8) one obtains what is called an equation of motion of third kind (Infeld and Plebanski, 1960):

(11)
$$\overline{\Omega} = 0.$$

Here and in the rest of this section a bar above a function of ψ will signify that ψ is to be replaced by the right-hand side of (10). Clearly, $\overline{\Phi} \equiv 0$. The equations (11) contain only the coordinates of the particle and one may enquire as to whether they can be deduced from a "mechanical" lagrangian.

Put
$$L(\xi, \dot{\xi}, \ddot{\xi}) = \overline{\Lambda} + \int_V \overline{\mathcal{L}}\, dx$$

and
$$\overline{W} = \int_{t_1}^{t_2} L\, dt.$$

Let $\delta\xi$ be a variation of the ξ's, vanishing at t_1 and t_2; the corresponding variation of the solution is

$$\delta\psi = \frac{\partial\Psi}{\partial\xi}\delta\xi + \frac{\partial\Psi}{\partial\dot{\xi}}\delta\dot{\xi}$$

and need not vanish on the boundary FV of V. A straightforward computation gives

$$\delta\overline{W} = \int_{t_1}^{t_2} dt \left(\overline{\Omega}\,\delta\xi + \int_V \overline{\Phi}\,\delta\psi\, dx + \int_{FV} \overline{\frac{\partial\mathcal{L}}{\partial\psi'}}\,\delta\psi\, dS \right.$$

where the notation is self-explanatory. One thus obtains a sufficient condition for the existence of a variational principle

(12)
$$\int_{FV} \overline{\frac{\partial\mathcal{L}}{\partial\psi'}}\,\delta\psi\, dS = 0 \Rightarrow \left(\delta\overline{W} = 0 \Leftrightarrow \overline{\Omega} = 0 \right).$$

Since $\overline{\Omega}$ contains no derivatives of ξ of order higher than the second, if the surface integral (12) vanishes then the dependence of L on $\ddot{\xi}$ is inessential.

The quantity
$$E = -L + \frac{\partial L}{\partial\dot{\xi}}\dot{\xi} - \frac{\partial L}{\partial\ddot{\xi}}\ddot{\xi}$$

is conserved if (12) holds and may be interpreted as the total energy of the system. In general

(13)
$$\frac{dE}{dt} = -\oint_{FV} S_i{}^k \dot{\xi}^i n_k\, dS$$

where
$$S_i{}^k = \overline{\frac{\partial\mathcal{L}}{\partial\psi_{,k}}}\frac{\partial\Psi}{\partial\xi^i} - \frac{d}{dt}\left(\overline{\frac{\partial\mathcal{L}}{\partial\psi_{,k}}}\frac{\partial\Psi}{\partial\dot{\xi}^i} \right)$$

The quantity
$$\oint_{FV} S_i{}^k n_k\, dS$$

particle. For simple systems the amount of radiation computed on the basis of (13) is in agreement with that obtained by other methods.

6. The method of asymptotic expansion.

To give a satisfactory and convincing theoretical answer to the problem of gravitational radiation it would be necessary to produce an exact or meaningfully accurate, solution of Einstein's equations for a realistic distribution of matter. Moreover, one would have to show that the corresponding material system undergoes secular changes which may be blamed on the gravitational waves emanating from the system. This has been too hard a task for anyone until now. In particular it is very difficult to find interesting and realistic non-static solutions of the interior problem and to connect them to appropriate exterior fields. However, a number of important global properties of material system, such as its mass or total momentum, may be recognised from a distance. Also, to calculate the radiated power by means of a Poynting vector it is sufficient to know the field in the far-away region. The recognition of this fact has been the starting point of a number of investigations on the asymptotic behaviour of gravitational fields.

The first step consisted in a rough formulation of Sommerfeld's radiation conditions for the gravitational field (Fock, 1955; Trautman, 1958c). By analogy with electrodynamics one may require that the Riemannian space-time V_4 admits a coordinate system x^μ, a null diverging vector field k^μ and a parameter r along the trajectories of this field, such that

(14) $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} = O(1/r),$$

(15) $$g_{\mu\nu,\rho} = i_{\mu\nu} k_\rho + O(1/r^2), \qquad i_{\mu\nu} = O(1/r),$$

(16) $$k_\mu = O(1), \qquad k_{\mu,\nu} = O(1/r),$$

and

(17) $$\left(i_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma} i_{\rho\sigma}\right) k^\nu = O(1/r^2).$$

Eq. (17) signifies that the coordinates are asymptotically harmonic; (15) implies the radiation condition: $g_{\mu\nu,\rho} k^\rho = O(1/r^2)$.

A coordinate transformation of the form

(18) $$x^\mu \longrightarrow x^{\mu'} + a^\mu,$$

(19) $$a_{\mu,\nu} = b_\mu k_\nu + O(1/r^2), \quad b_\mu = O(1/r)$$

preserves (14), (15) and (17). Indeed, it follows from (19) that

$$a_{\mu,\nu\rho} = c_\mu k_\nu k_\rho + O(1/r^2), \quad c_\mu = O(1/r)$$

and (18) induces the transformation

(20) $$i_{\mu\nu} \rightarrow i'_{\mu\nu} = i_{\mu\nu} + c_\mu k_\nu + c_\nu k_\mu .$$

In the asymptotic region, the pseudotensor of momentum and energy becomes

$$t_\mu{}^\nu = \tau k_\mu k^\nu + O(1/r^3)$$

where

$$\tau = \frac{1}{32\pi k} i^{\mu\nu}\left(i_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma} i_{\rho\sigma}\right).$$

As a consequence of (17), τ is never negative and is invariant under the coordinate transformations (20). It is possible, therefore, to obtain the total radiated energy and momentum by computing a suitable integral of t_μ^ν. Recently, Cornish (1964) has shown that the value of that integral does not depend on which expression is taken for t_μ^ν, within a wide class of energy-momentum pseudotensors, provided that the boundary conditions (14) - (17) are satisfied. In an earlier paper, Komar (1962) found a formulation of the boundary conditions more satisfactory from the point of view of geometry than those reviewed here. He was able to express these conditions in terms of asymptotic Killing fields. He also showed that the transformations (20) may be used to achieve

$$i_{\mu\nu} k^\nu = O(1/r^2)$$

in addition to (17). According to Komar, these more stringent boundary conditions make it possible to apply Møller's expression for energy and momentum to radiative spaces (Møller, 1958).

The asymptotic form of the curvature tensor may also be obtained from the boundary conditions. One gets

$$i_{\mu\nu,\rho} = j_{\mu\nu} k_\rho + O(1/r^2), \quad j_{\mu\nu} = O(1/r),$$

and $$\left(j_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma} j_{\rho\sigma}\right) k^\nu = O(1/r^2)$$

(21) $$R_{\mu\nu\rho\sigma} = \tfrac{1}{2} k_{[\mu} j_{\nu][\rho} k_{\sigma]} + O(1/r^2).$$

The $1/r$ part of the Riemann tensor is algebraically of type null. This means that the $1/r$ part of the field equations is automatically satisfied because of the boundary conditions imposed on the metric.

To obtain more detailed information about the physics and geometry of the radiative space, it would be necessary to solve the field equations with greater accuracy. The present formulation is little suitable for the purpose. One may say that the reason for this being so is that not enough geometrical elements have been adapted to the problem.

Let the propagation vector of the waves be hypersurface-orthogonal; when properly normalized it will be a gradient vector, $k_\mu = u_{,\mu}$. If ξ and η are two coordinates constant along the trajectories of k^μ, the line-element of V_4 is

$$(22)\quad ds^2 = -P^2\left[e^\alpha \cosh\beta\,(d\xi - A\,du)^2 + 2\sinh\beta\,(d\xi - A\,du)\times(d\eta - B\,du) + e^{-\alpha}\cosh\beta\,(d\eta - B\,du)^2\right] + C\,du^2 + 2D\,du\,dr,$$

where P, α, β, A, B, C and D are functions of the coordinates ξ, η, r and u (Robinson, 1962); Sachs, 1962). This is a completely general line-element; if one demands that k^μ be shear-free and chooses r to be an affine parameter along the trajectories of k^μ, which are now null geodesics (= rays), then $\frac{\partial\alpha}{\partial r} = \frac{\partial\beta}{\partial r} = 0$, D = 1, and by a coordinate transformation the form (22) may be reduced to (Robinson and Trautman, 1964)

$$(23)\quad ds^2 = -P^2\left[(d\xi - A\,du)^2 + (d\eta - B\,du)^2\right] + C\,du^2 + 2\,du\,dr.$$

It is known, however, that only very special metrics admit non-shearing congruences of null geodesics (cf. section 8). None of the metrics (23) describes a realistic kind of gravitational radiation.

Bondi was the first to consider, in a systematic manner, a metric general enough to represent radiation from a bounded source (Bondi, 1960). He kept the shear but restricted the field to be axially symmetric. Sachs (1962) showed that nothing essential was lost by the latter assumption. An important feature in Bondi's approach is his choice of r: he takes for it the luminosity distance, more precisely, for Bondi's line-element

$$(24)\quad ds^2 = -r^2\left[e^\alpha (d\theta - A\,du)^2 + e^{-\alpha}\sin^2\theta\,d\varphi^2\right] + C\,du^2 + 2D\,du\,dr,$$

the area of the surface u = const., r = const., $0 \leq \varphi < 2\pi, 0 \leq \theta \leq \pi$ is $4\pi r^2$. Bondi assumes that for sufficiently large values of r the functions α, A, C and D have the form*

*The lines to follow contain a grossly simplified and mutilated presentation of the work by Bondi et al., (Bondi, van der Burg and Metzner, 1962).

$$
\begin{aligned}
\alpha &= \frac{n}{r} + O(1/r^2), \\
A &= \frac{a}{r} + O(1/r^2), \\
C &= 1 - \frac{2m}{r} + O(1/r^2), \\
D &= 1 + d/r + O(1/r^2),
\end{aligned}
\tag{25}
$$

where a, m, n, d are functions of u and θ only. These assumptions imply that (24) may be transformed to an asymptotically Cartesian coordinate system in which the radiation conditions (14) - (17) will be satisfied. Bondi shows that the expansions (25) are consistent with the fiald equations for empty space and proceeds to solve some of these equations to obtain a = d = 0 and a relation between m and n,

$$
\frac{\partial m}{\partial u} = -\left(\frac{\partial n}{\partial u}\right)^2 + \frac{1}{2}\frac{\partial}{\partial u}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(n\sin^2\theta\right). \tag{26}
$$

There are also other equations relating m and n to higher order terms but they will not be considered here. It appears from the analysis that n may be a completely arbitrary function of its arguments; its derivative with respect to u is called the news function. The function m is closely related to the total energy of the system. In the static case, $\frac{\partial m}{\partial u} = 0$ and from other field equations also $\frac{\partial m}{\partial\theta} = 0$ and m may be identified with the mass. In general, Bondi defines the mass as the average of m over the angles,

$$
M(u) = \frac{1}{2}\int_0^\pi m(u,\theta)\sin\theta\, d\theta.
$$

Equation (26) implies that the mass diminishes,

$$
\frac{dM}{du} = -\frac{1}{2}\int_0^\pi \left(\frac{\partial n}{\partial u}\right)^2 \sin\theta\, d\theta
$$

if and only if there is news. Clearly, $\frac{\partial n}{\partial u}$ plays the role of what was previously denoted by $i_{\mu\nu}$. The Riemann tensor is of the general form (Sachs, 1962)

$$
R = \frac{N}{r} + III/r^2 + D/r^3 + \dots \tag{27}
$$

where th[illegible]pressed and N, III and D denote tensors [illegible] and degenerate, respectively (cf section 7). These tensors admit k_μ as a principal null vector and are covariantly constant along the rays; moreover they are proportional to quantities which may be given a physical meaning:

$$N \simeq \partial^2 r/\partial u^2,$$
$$III \simeq \frac{\partial^2}{\partial u \partial \theta} r \sin^2\theta,$$
$$D \simeq 2m + \frac{\partial r^2}{\partial \theta}.$$

Only the first of these results could have been obtained from (21). Bondi insists on the relation between news and loss of mass and, on the basis of it, advances the hypothesis that there is no radiation from freely falling particles and a pressure-free dust.

A different technique to solve the field equations has been developed by Newman and Penrose (1962). They introduce a field of null basis vectors (tetrads) adapted to a congruence of diverging, hypersurface-orthogonal rays in V_4. If k_μ is tangent to the rays, the null tetrad is $(k_\mu, l_\mu, m_\mu, \overline{m}_\mu)$. The vectors are normalized so that $k_\mu l^\mu = 1 = -m_\mu \overline{m}^\mu$ and the remaining scalar products are zero. Given such a tetrad, the Riemann tensor of an empty space-time may be split according to the formula

(28) $$R = N(k) + III(k) + D(k,l) + III(l) + N(l).$$

Here N(k) denotes a tensor of Petrov type null, admitting k_μ as a propagation vector; D(k,l) is a degenerate tensor and as such admits two principal null vectors, k_μ and l_μ (cf section 7). Newman and Penrose introduce as one of the coordinates the affine parameter r along the rays and replace the Einstein equations by a set of equations of the first order on the Ricci rotation coefficients corresponding to the null tetrad. Roughly, their result on the asymptotic behaviour of the Riemann tensor in empty space is:

if $N(l) = O(1/r^5)$, then
$$III(l) = O(1/r^4),$$
$$D(k,l) = O(1/r^3),$$
$$III(k) = O(1/r^2),$$

This is in agreement with (27) and also with earlier exact results on the behaviour of the curvature tensor for algebraically special metrics (Sachs, 1961; cf section 8).

Another important result on the asymptotic behaviour of gravitational waves is the "wave-front theorem", which in different forms was independently demonstrated by Papapetrou (1958), Peres and Rosen (1959), Infeld and Plebanski (1960) and Misner (1962). Papapetrou has shown that there can be no periodic, non-static and asymptotically Euclidean metrics; possible are only pulse waves, with the amplitude of the disturbance decreasing sufficiently rapidly for $t \rightarrow \pm\infty$. Infeld and Plebanski prove that the assumptions

(29) $\quad g_{\mu\nu} = \eta_{\mu\nu} + O(1/r),$

(30) $\quad g_{\mu\nu,\rho} = O(1/r)$ but not $O(1/r^2)$, for t = const.

lead to contradictions with the field equations. Very roughly, their argument is as follows: suppose that on some space-like Hypersurface t = const., and for a certain choice of coordinates, the components of the metric tensor are such that (29), (30) hold. For some relevant components of $g_{\mu\nu}$ the field equations can be symbolically written as

$$\Delta\varphi = \text{const.}\,(\nabla\varphi)^2 .$$

If the right hand side behaves really like $1/r^2$, φ contains a log r term in contradiction to (29).

Arnowitt, Deser and Misner formulate a stronger form of the wave-front theorem and prove it rigourously: if (29) holds, then for each t = const. one can further restrict the coordinates so that

$$g_{ij,k} = 0(1/r^{\frac{3}{2}+\epsilon})$$
$$K_{ij} = 0(1/r^{\frac{3}{2}+\epsilon})$$

where $\epsilon > 0$ and K_{ij} is the second fundamental form of the hypersurface t = const. (Misner, 1962).

Geometry of null elements

It is easy to convince oneself about the intimate connection between waves and null elements in space-time. There are obvious physical reasons for such a connection; electromagnetic and gravitational waves travel with the velocity of light; in four dimensions, a ray of light is a null geodesic; the electromagnetic tensor of a plane, progressive wave is $k_{[\mu} m_{\nu]}$, with k^{μ} null; the Cauchy problem cannot be locally formulated on a null hypersurface because waves may whizz past without consideration for the value of the field on the hypersurface. The possibility of transferring information by waves is also related to the null character of the geometric structures associated with them. The role played by the null elements is already apparent in the study of asymptotic properties of radiative fields. During the recent years, a large number of important papers have been written on Riemannian space-times with distinguished null structures. There are very readable expositary articles on this subject (Pirani, 1962, 1964; Sachs, 1963; Ehlers and Kundt, 1962; these works contain an extensive bibliography). It does not seem necessary to review here these papers in detail. The following sections contain a very brief enumeration of some of the important results.

7. Algebra of the conformal tensor.

Pirani's research on the physical meaning of the Petrov classification of the curvature tensors may be considered as the starting point of the recent trend to use geometrical methods in gravitational radiation theory. (Petrov, 1954; Pirani, 1957). The Petrov classification itself has been the object of numerous studies and presentations. An account of the various approaches and an extensive bibliography of the subject may be found in the review article by Pirani (1962). For our purposes it suffices to recall the spinor approach (Penrose, 1960; see also Witten, 1959): there is a one-to-one correspondence between directions in spinor space (i.e., in C^2) and null directions in Minkowski vector space: every real tensor belonging to an irreducible may be represented by a symmetric 2s-index spinor $\varphi_{AB\ldots K}$; every such spinor may be factorized,

$$\varphi_{AB\ldots K} = \xi_{(A}\eta_{B}\cdots\chi_{K)}$$

Therefore, any non-zero tensor of the type $\mathcal{D}(s,0)\oplus\mathcal{D}(0,s)$ defines 2s null directions, called principal, some of which may coincide. A multiple principle null direction is called a propagation direction. The Petrov classification, as formulated by Penrose, consists in enumerating all possible coincidences among the principal null directions. The Riemann tensor in empty space, and the Weyl tensor of conformal curvature in any case, correspond to s = 2. They define 4 null directions. One calls a conformal tensor type I if all these directions are distinct; type II, III or N (null) if exactly two, three or four coincide, type D if there are two distinct pairs of coinciding principal null directions. A space is called algebraically special if its Weyl tensor is not of type I. For the electro-magnetic field, s = 1, and there are only two types of tensors.

Plane waves and similar simple kinds of radiation are of type null; the Schwarzschild metric is D; it is quite clear that a general, physically realistic space-time is of type I.

8. Local differential properties.

An algebraically special, conformally non-flat metric defines a preferred field of null directions - the propagation directions (a D metric defines two such fields). This field of directions defines in turn a congruence of rays in space-time. One may think of these rays as corresponding to a special pattern of propagation of light (Robinson, 1961). In special cases, such as that of plane waves, one may interpret the repeated principal null directions as the directions of propagation of gravitational radiation. These remarks justify the interest people have recently shown in the properties of rays, in particular of those associated with algebraically special metrics.

geodesics, not necessarily associated with principal null directions. It is possible to normalize the tangent vectors so that

$$k_{\mu;\nu} k^{\nu} = 0.$$

From the first derivatives of k_{μ} one can then form exactly three scalars: the coefficient of rotation,

$$\omega = \sqrt{\tfrac{1}{2} k_{[\mu;\nu]} k^{\mu;\nu}},$$

the divergence,

$$\theta = \tfrac{1}{2} k^{\mu}{}_{;\mu},$$

and the shear,

$$\sigma = \sqrt{\tfrac{1}{2} k_{(\mu;\nu)} k^{\mu;\nu} - \theta^2}.$$

These quantities can be given a simple optical interpretation (Sachs, 1961): think of the null geodesics as of rays of light. Consider a small, plane, opaque object and a plane screen, some distance apart from the object. Suppose that both the object and the screen are oriented so that they are orthogonal to the rays of light in their respective rest frames and situated so that the shadow cast by the object can be observed on the screen. One can displace, by parallel transport along the rays, the object to the position occupied by the screen and compare it with the shadow. The magnification of the shadow is proportional to θ, the rotation - to ω and σ characterizes the shear (deformation).

The following theorem is due to Goldberg and Sachs (1962): a vacuum metric is algebraically special if and only if it contains a shear-free congruence of rays; the tangent vector to the congruence belongs to a propagation direction of the conformal curvature tensor. A stronger form of this theorem was established by Robinson and Schild (1962).

Sachs integrated some of the field equations in empty space to obtain the exact behaviour of algebraically special Riemann tensors along the rays (Sachs, 1961 a, b). Only one of his results is quoted here: for an algebraically special, empty space-time with $\omega = 0 \neq \theta$ the Riemann tensor is

(31) $$R = N/r + \underline{\underline{III}}/r^2 + \underline{\underline{II}}/r^3.$$

Here N, III, and II are tensors of the type indicated by the parameter. It has been possible to strengthen this result by proving that II must be of type D and finding the precise form of the line-element (Robinson and Trautman, 1962).

It is interesting to compare the exact result (31) with an analogous formula obtained by approximate methods (section 6). The coincidence in form of the three leading terms in (27) and (31) may be interpreted to mean that certain algebraically special fields constitute good approximations to actual radiation fields at large distances from the source. Still, these fields are too special to be realistic themselves.

9. Null elements at infinity.

A beautiful method for dealing with the asymptotic problems for mass-zero fields has been recently developed by Penrose (1962, 1963, 1964). Leaving out the subtleties, the essential ideas underlying his approach may be summarized as follows. When one talks about the 'infinity' of a certain topological space, one has in mind that the space, though non-compact is locally compact and can be compactified by adjunction of certain ideal (infinite) elements. There are many ways of compactifying a locally compact space; some of them may be preferred if the space possesses other structures, in addition to its topology. For example, the Minkowski space can be compactified so that its conformal geometry can be extended, by continuity, to the infinite elements. Furthermore, the equations of motion of mass-zero fields are conformally invariant in the sense that, given two Riemannian space-times conformally related to each other and a solution of a mass-zero field equation in one of them, there is a natural way of mapping the solution into a solution of the same equation in the other space. According to Penrose, instead of considering the asymptotic behaviour of mass-zero fields in non-compact Riemannian spaces, one can look at their properties in neighbourhoods of certain elements of a compact space, conformal to the given one.

The construction of the Penrose manifold $\mathcal{P}$ for a Minkowski space M is straightforward. Consider ... $^{-4}$

and introduce in each of them coordinates u, v, θ, φ, where

$$u = t-r, \quad v = t+r$$

and r, θ, φ are the spherical coordinates. Let one of the R^4's be given the Minkowski structure; this space will be called $\mathcal{M}$. Consider, in the second space, the subset defined by

$$-\pi/2 < u \leq v < \pi/2$$

Its closure in R^4 is compact, call it $\mathcal{P}$. The transformation

$$u \longrightarrow \text{arc tan } u,$$

$$v \longrightarrow \text{arc tan } v,$$

$$\theta \longrightarrow \theta, \quad \varphi \longrightarrow \varphi$$

is a homeomorphism of $\mathcal{M}$ onto the interior of $\mathcal{P}$; it is also a diffeomorphism of the natural differentiable structures. The metric form transported by this diffeomorphism from $\mathcal{M}$ to Int $\mathcal{P}$ is

(32) $$\frac{2\,du\,dv}{\cos^2 u \cos^2 v} - \frac{1}{4}(\tan v - \tan u)^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Let us weaken the geometry in Int $\mathcal{P}$ by taking the conformal geometry induced by (32); another metric form representing the same geometry is $$2\,du\,dv - \frac{1}{4}\sin^2(u-v)(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

This may be extended by continuity over the whole $\mathcal{P}$. To obtain a non-degenerate conformal geometry on $\mathcal{P}$ it is necessary to introduce there an equivalence relation: one identifies with each other points for which (simultaneously) $u = -\pi/2$, $v = \pi/2$ (the spatial infinity). The 'hypersurfaces at infinity' $u = -\pi/2$ and $v = \pi/2$ are null in the extended conformal geometry.

The construction of $\mathcal{P}$ is not so simple in the case of a Riemannian manifold even if its topology is Euclidean. Even in simple cases, the conformal geometry may be inextendable over the entire $\mathcal{P}$. Such singularities occur, e.g., for the Schwarzschild space-time. When the cosmological constant is non-zero, the hyper-surfaces at infinity cease to be null.

With the aid of the conformal technique, Penrose was able to give a very simple proof of a general theorem on the asymptotic form of mass-zero fields (the peeling-off theorem). His method is suitable for treating the groups of a symptotic symmetries and yields a new approach to the problem of gravitational energy.

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