Short Summary

of the lectures on

Theory of Gravitational Waves and Radiation

prepared by Andrzej Trautman/Warsaw/

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Introduction: a short historical account. Analogies between gravitation and electromagnetism. Gravitational radiation according to the linear theory. The problem of energy of the gravitational field. Sommerfeld's radiation conditions in the theory of gravitation. The equation of geodetic deviation and the detection of gravitational waves. Optics in a Riemannian space-time. Algebraically degenerate fields and exact wave-like solutions of Einstein's equations. Approximate methods of solving the field equations and treating the problem of radiation. Gravitational radiation reaction force.

Reccomended literature

Y. Choquet-Bruhat, Géométrie différentielle et systèmes extérieurs, Dunod, Paris, 1971 /geometry/;

L.D. Landau and E.M. Lifshitz, The classical Theory of Fields, Pergamon Press, London 1959/standard text/;

J. Weber, General Relativity and Gravitational Waves, Interscience, New York, 1961 /short survey of relativity theory with a description of the famous Weber detectors/;

C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation, Freeman, San Francisco, 1973 /modern book with much material on radiation/;

Gravitational Radiation and Gravitational Collapse, ed.by. C. DeWitt-Morette, D. Reidel, Dordrecht, 1974 /recent experimental results on the detection of gravit. waves/;

Ondes et radiations gravitionnelles, Colloque du CNRS, Paris, 1974 /recent theoretical and experimental results/

Introduction to the Theory of Gravitational Radiation
(Selected Problems)

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Introduction

These notes are based on the lectures which I gave ~~given by the author~~ at the International School of Cosmology and Gravitation (4th Course: Gravitational Waves) held from 13 to 25 March 1975 at the "Ettore Majorana" Centre for Scientific Culture in Erice (Sicily). I thank Professors A. Zichichi, N. Dallaporta, V. De Sabbata, L. Gratton and J. Weber for their invitation to deliver the lectures and ~~to the~~ the "Ettore Majorana" Centre for financial support. ~~which made~~

The notes reflect the informal character of the lectures. They were written in Erice, where I had no access to a library which would allow me to provide the appropriate references. In part, the notes are based on my earlier lectures and papers, which appear to be sufficiently inaccessible to justify a ~~partial~~ repetition of the material contained in them. The lectures

read during the first days of the school and meant
were ~~given at the beginning of the course, as~~
as an elementary introduction to the theory
of gravitational waves, intended mainly for those
participants who so far had had little knowledge
of the subject. Many fundamental and important
problems have been omitted because they
are presented in detail in other lectures
given at the School.

At the beginning of a course such as
this, it seems appropriate to ask the question:
Why is there now so much interest in the phenomenon
of gravitational radiation? ~~[illegible]~~ One can
answer it simply by saying that, thanks to the
pioneering work of Joseph Weber and the
efforts of many research groups throughout the
world, ~~[illegible]~~ there are reasonably good prospects
~~[illegible]~~ for detecting gravitational waves in not-too-distant a future. Such ~~an answer raises~~
a statement raises more questions than it answers:
why are we so keen on detecting the waves,
how will a success in this field affect our
~~understanding of physics~~ theoretical knowledge,
and what further progress can we expect ~~to derive~~
~~to follow~~ ~~from~~ from this achievement. Concerning the first question,

I believe an honest answer is to say that we want to solve the problem because it is there - it is a challenge similar to that of climbing a high and steep mountain. As soon as we detect gravitational radiation, the correctness of our general ideas on ~~about~~ the nature of relativistic gravity will be confirmed. It is unlikely, however, that in the near future we shall be able to use experiments on gravitational waves to discriminate between various ~~rival theories~~ tensor theories of gravity; at the best, we can expect to ~~make~~ obtain some estimates of the scalar component of radiation predicted by some of the modifications of the Einstein theory [1]. In a somewhat more distant future, gravitational waves will provide an ~~basis for~~ ~~tool~~ important, new tool for exploring the Universe. Gravitational wave astronomy [2] will allow us to look at unusual phenomena in the Universe, ~~characterized by~~ ~~accompanied~~ characterized by violent motions and compact sizes of the masses in question, but not necessarily by (the production of) large amounts of electromagnetic radiation. Finally, I should say that the detection of gravitational waves will provide an additional stimulus for the search for a satisfactory quantum theory of gravity. It is hard to imagine that ~~gravitational waves can interact with atomic~~ ~~sy~~ the interaction of gravitational waves with matter on the microscopic level can be very different from that of electromagnetic radiation with atomic systems.

Gravitational radiation for pedestrians

Many of our ideas on gravitational radiation are based on analogies between electromagnetism and gravity. The similarity between the Coulomb and Newton laws, or between the Poisson equations underlying both electrostatics and the ~~Newton~~ non-relativistic theory of gravitation, is so close as to be somewhat misleading. The attractive character of gravitational forces and the identity of gravitational charges with the inertial masses warn us against taking the analogy too seriously. An accurate, ~~qualitative~~ quantitative description of gravitational radiation may be obtained only within the framework of a well-defined theory of gravitation, by means of rather subtle computations which take into account the intrinsic non-linearity of the field equations and ~~the necessity of adapting~~ are based on using approximati methods adapted to the physical situation under consideration. A detailed (and sophisticated) survey of such methods, with a discussion of their validity within the framework of Einstein's theory of gravitation, has been presented at the School by Kip Thorne and Charles Misner. In this lecture, I present a very simple-minded derivation of the basic formula for the amount of radiation produced by non-relativistic systems and show how it can be used to estimate the order of magnitude of gravitational radiation. ~~emanating from simple systems.~~

Consider first a system of electric charges described within the framework of special relativity by a density $\rho(\vec{r}, t)$. The wave equation for the scalar potential may be solved to give

$$(1) \qquad \varphi(\vec{r}, t) = \int \frac{1}{R} \rho(\vec{r}\,', t - R/c)\, d_3x'$$

where

$$R = |\vec{r} - \vec{r}\,'|$$

and c is the velocity of light. Let v, L, and λ denote, respectively, a typical velocity of the charges, the linear dimension of the region where they move, and a typical wavelength of the electromagnetic radiation produced by the charges. For a non-relativistic system, $v \ll c$ and $L \ll \lambda$, so that R ~~may be insig~~ ~~repla~~ occurring in the argument of ρ may be

In the wave zone, where $r \gg \lambda$, the distance R in the denominator ~~and the~~ of the integrand may be replaced by r.

replaced by $r - \vec{r}'\vec{r}/r$ and ρ expanded into a power series in $1/c$,

$$(2) \qquad \varphi = \frac{e}{r} + \frac{\vec{r}\cdot\dot{\vec{d}}(t-r/c)}{cr^2} + \dots$$

where

$$e = \int \rho \, d_3x$$

is the total charge, and

$$\vec{d}(t) = \int \vec{r}\,\rho(\vec{r},t)\, d_3x$$

is the dipole moment. Although the scalar potential φ does not tell the full story about the electromagnetic field – important information ~~is~~ being contained in the vector potential – eq. (2) happens to be sufficient to estimate the order of magnitude of the ~~amount of~~ electromagnetic ~~...~~ energy radiated in the dipole approximation. ~~radiation~~ The electric field is of the order of $|\mathrm{grad}\,\varphi| \sim \ddot{d}(t-r/c)/cr + O(1/r^2)$. In the wave zone, the electric ~~field~~ and magnetic fields are approximately equal. This information is sufficient to give the estimate $P_e \sim \ddot{d}^2/c^3$. The precise formula for the energy radiated per unit time in the form of (electric) dipole waves is known to be

$$(3) \qquad P_e = 2\ddot{\vec{d}}^2/3c^3 .$$

By analogy with electromagnetism, let us consider a wave equation

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 4\pi G \varrho$$

for the gravitational scalar potential φ due to a distribution of masses with density ϱ; G is the gravitational constant. Under similar assumptions as in the electromagnetic case, we arrive at the multipole expansion

$$(4) \quad \varphi = -G\left(\frac{m}{r} + \frac{\vec{r}\,\vec{p}}{c r^2} + \frac{x^i x^j \ddot{Q}_{ij}(t-r/c)}{2c^2 r^3} + \dots\right)$$

where

$$m = \int \varrho \, d_3 x$$

is now the total mass,

$$Q_{ij} = \int (x_i x_j - \tfrac{1}{3}\delta_{ij} r^2)\varrho \, d_3 x$$

is the tensor of quadrupole moment $(i, j = 1, 2, 3)$, and the equation of continuity

$$\dot{\varrho} + \operatorname{div}(\varrho \vec{v}) = 0$$

has been used to show that

$$\vec{p} = \int \dot{\varrho}\, \vec{r} \, d_3 x$$

is the total momentum of the system. The three dots in (4) stand for the higher multipoles and also for the term corresponding to the spherically symmetric second moment $\sim \int r^2 \rho \, d_3x$. Because

~~[illegible]~~

both mass and non-relativistic momentum are conserved in the Newtonian limit, we do not expect gravitational radiation to contain simple monopole or 'electric' dipole terms. ~~begin with the electric quadrupole term.~~

~~[illegible]~~

From the observations of the bending of light rays passing near the surface of the Sun, we know the gravitational field to have tensor character. Therefore, similarly as in electrodynamics, the scalar potential (4) contains only a part of the information about the gravitational field. For example, it tells nothing about the gravitational effects due to the rotation of ~~masses~~ bodies. These effects are analogous to those connected with ~~the~~ a stationary magnetic field. Since total angular momentum is conserved, we cannot expect to have, ~~grav~~ in the low-speed limit, gravitational radiation of the 'magnetic' dipole type. The spherically symmetric moment $\int r^2 \rho \, d_3x$ would lead to a 'scalar' wave, corresponding to spin zero gravitons. They are admitted in some of the modifications of the Einstein theory, such as the Brans-Dicke theory. The evidence based on the deflection of light is against the scalar admixtures and we shall neglect them altogether. This leaves us with the quadrupole term as (providing) the main contribution to the power radiated by a system of bodies in slow motion. By a simple dimensional analysis we get from (4):

$$P_g \sim \frac{c}{G}\oint (\nabla\varphi)^2\, dS \sim \frac{G}{c^5}\,\dddot{Q}_{ij}^{\,2}$$

The missing coefficient may be supplied by a heuristic argument based on the magnitude of the spin of gravitons or by having ~~a heuristic arg~~ recourse to standard formulae of a linear theory for a second rank tensor field,

(5) $$P_g = \frac{G}{5c^5}\,\dddot{Q}_{ij}^{\,2}$$

As crude as the above arguments may appear to be, they lead to a formula for the radiated power which is correct ~~in the~~ for nearly Newtonian systems ~~in a~~ described according to any relativistic theory which reduces to a spin-two, mass-zero theory in the limit of weak fields (cf. K. Thorne's lectures).

Formulae (3) and (5) may be used to compare the amount of energy radiated in the form of electromagnetic and gravitational waves by ~~a syst~~ a nearly circular motion of ~~a system of~~ two opposite charges, e and -e, and ~~a system~~ of two equal masses m, respectively.

If we are interested only in the orders of magnitude, then, denoting by a and ω the radius of the circle and the angular velocity of the particles, we may write

$$d \sim ea, \quad \ddot{d} \sim ea\omega^2, \quad m\omega^2 a \sim e^2/a^2$$

(6) $$P_e \sim mc^2\omega\left(\frac{e^2}{mc^2 a}\right)^{5/2}$$

~~in the for electromagnetic radiation, and~~
in the electromagnetic case, and

$$Q \sim ma^2, \quad \dddot{Q} \sim ma^2\omega^3, \quad m\omega^2 a \sim Gm^2/a^2$$

(7) $$P_g \sim mc^2\omega\left(\frac{Gm}{c^2 a}\right)^{7/2}$$

for particles moving under their mutual gravitational attraction. In each case the amount of radiated power depends strongly on ~~the~~ a dimensionless parameter,

$$\frac{e^2}{mc^2 a} \quad \text{or} \quad \frac{Gm}{c^2 a}.$$

Their ratio, e^2/Gm^2, is of the order of 10^{42} for electrons. For an atom, $e^2/mc^2 a \sim$ $\sim e^4/\hbar^2 c^2 \sim 1/(137)^2$, whereas the ratio $Gm/c^2 a$ for 'normal' binary stars is very small indeed [ ]. Clearly, the ~~higher~~ exponent in (7) is higher than in (6) because of the quadrupole character of gravitational radiation.

The quadrupole-moment formula (5) is not applicable to relativistic phenomena such as the emission of gravitational radiation by ~~accretion of~~ matter accreting on black holes. It may be sometimes used, however, to get a very rough estimate of the factors determining radiation even in relativistic situation. Consider, for example, the case of a particle of mass m falling radially onto a black hole of ~~#~~ mass M. If the particle has ~~t~~ no kinetic energy at large distances, ~~from the~~ then, according to the Newtonian theory,

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = 0$$

This may be solved to give

$$r(t) = \left(\frac{9}{2} GMt^2\right)^{1/3}$$

Since the quadrupole moment Q is of the order of mr^2, ~~we may get the or~~ and no radiation will reach an external observer from ~~after~~ the particle ~~has~~ ~~will~~ after it will have reached the Schwarzschild radius

$$2GM/c^2 = r(t_g), \quad t_g = 4GM/3c^3,$$

we may evaluate the order of magnitude of energy radiated to be

$$(8) \qquad E \sim \frac{G}{5c^5} \int_{-\infty}^{t_g} \dddot{Q}^2 \, dt = \frac{1}{35} mc^2 \frac{m}{M}$$

This is about one order of magnitude ~~too~~ more than the value obtained by a precise computation [ ]. The reason for the discrepancy is ~~connected to~~ due, in part, to our having neglected the ~~red shift~~ gravitational and Doppler red shifts of the radiation emitted by the particle. The simple-minded computation exhibits, however, the factor m/M which shows that radiation is small if $m \ll M$. The importance and generality of this result is discussed at length in the lectures of R. Ruffini.

By inspection of formulae (7) and (8) we may predict that a significant fraction of the rest mass ~~a~~ may be radiated away in the form of gravitational waves only by systems of comparable masses moving at distances ~~from one another~~ ~~to~~ from one ~~a~~ another which are of the order of their Schwarzschild radii.

Gravitational energy

In general relativity, the notions of energy, momentum and angular momentum are not such good concepts as they are in special relativity. Gravitational energy cannot be precisely localized. This is essentially due to the principle of equivalence: in classical physics, the concept of energy is inseparable of that of force. The principle of equivalence says that (at least locally) gravitational forces cannot be meaningfully defined.

In many special cases it is possible, however, to define ~~a~~ ~~total~~ conserved quantities corresponding to the total content of energy, momentum, etc., of a material, gravitating system. These quantities may be also used to evaluate the amount of ~~a~~ energy emanating (in the form of radiation) from bounded sources. For gravitational waves of short wavelength, ~~it is possible to~~ R. Isaacson defined an effective energy-momentum tensor. ~~Such closely~~ ~~A~~ Its properties closely resemble those of the Maxwell energy-momentum tensor (cf [ ] and the lectures of C. Misner).

In this lecture, I present a ~~very~~ simple derivation of the so-called von Freud 'superpotentials' and the Einstein energy-momentum 'pseudotensor' of the gravitational field and show how they can be used to compute the total energy-momentum and its rate of change.

It is convenient to use Cartan's calculus of differential forms [-]. At a point p of the four-dimensional Riemannian manifold X serving as a model of space-time, one introduces ~~a~~ a co-frame, which is a set (θ^μ) of four linearly indendent ~~one~~ 1-forms. For example, if (x^μ) are local coordinates around p, one may take the holonomic ~~co~~ co-frame by putting $\theta^\mu = dx^\mu$. In many cases it is convenient to use other co-frames, e.g., orthonormal or null. The Riemannian metric may be written as

$$g = g_{\mu\nu}\, \theta^\mu \otimes \theta^\nu$$

whereas the linear connection is described by a set of 1-forms $\omega^\mu{}_\nu$,

$$\omega^\mu{}_\nu = \Gamma^\mu_{\nu\rho}\, \theta^\rho .$$

The Riemannian character of the geometry is ~~expressed~~ guaranteed by the vanishing of ~~the~~

torsion,

(9) $$D\theta^\mu \underset{\text{def}}{=} d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0$$

and by the covariantly constant nature of the metric,

(10) $$Dg_{\mu\nu} \underset{\text{def}}{=} dg_{\mu\nu} - \omega_{\mu\nu} - \omega_{\nu\mu} = 0$$

The curvature is described by the collection of 2-forms

(11) $$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$$

or by the corresponding tensor, $R^\mu{}_{\nu\rho\sigma}$, defined by

$$\Omega^\mu{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma .$$

The Levi-Civita ~~tensor η~~ (pseudo)tensor $\eta_{\mu\nu\rho\sigma} = \eta_{[\mu\nu\rho\sigma]}$, $\eta_{0123} = |\det g_{\alpha\beta}|^{1/2}$, may be used to define a set of forms,

$$\eta_{\mu\nu\rho} = \theta^\sigma \eta_{\mu\nu\rho\sigma} ,$$

$$\eta_{\mu\nu} = \tfrac{1}{2} \theta^\rho \wedge \eta_{\mu\nu\rho} ,$$

$$\eta_\mu = \tfrac{1}{3} \theta^\nu \wedge \eta_{\mu\nu} ,$$

$$\eta = \tfrac{1}{4} \theta^\mu \wedge \eta_\mu .$$

the last of which is the 4-form of volume of X. If we use units such that $c = 1 = G$ and write $T_\mu = T_\mu{}^\nu \eta_\nu$, then Einstein's equations become

(12) $$\tfrac{1}{2} \eta_{\mu\nu\rho} \wedge \Omega^{\nu\rho} = -8\pi T_\mu$$

Using eq. (11), the ~~last~~ ~~ed~~ Einstein equation may be written as

(13) $$T_\mu + \frac{1}{16\pi}\eta_{\mu\nu}{}^{\rho}\wedge\left(d\omega^\nu{}_\rho + \omega^\nu{}_\sigma\wedge\omega^\sigma{}_\rho\right) = 0$$

or

(14) $$4\pi\left(T_\mu + t_\mu\right) - d\overset{4\pi}{U}_\mu = 0$$

where

(15) $$4\pi U_\mu \underset{\text{def}}{=} \frac{4\pi}{16\pi}\eta_{\mu\nu}{}^{\rho}\wedge\omega^\nu{}_\rho = \cancel{\ } U$$

is set of ~~4~~ four 2-forms, corresponding to the von Freud superpotential, $U_\mu{}^{\nu\rho}$,

(16) $$U_\mu = \frac{1}{2}U_\mu{}^{\nu\rho}\eta_{\nu\rho}$$

and

(17) $$t_\mu = t_\mu{}^\nu\eta_\nu .$$

The Einstein 'pseudotensor' of energy and momentum $t_\mu{}^\nu$ may be easily read off from (13) by taking into account eqs. (14), (15), (17), and the relation

$$D\eta_{\mu\nu}{}^{\rho} \underset{\text{def}}{=} d\eta_{\mu\nu}{}^{\rho} - \omega^\sigma{}_\mu\wedge\eta_{\sigma\nu}{}^{\rho} - \omega^\sigma{}_\nu\wedge\eta_{\mu\sigma}{}^{\rho} + \omega^\rho{}_\sigma\wedge\eta_{\mu\nu}{}^{\sigma} = 0$$

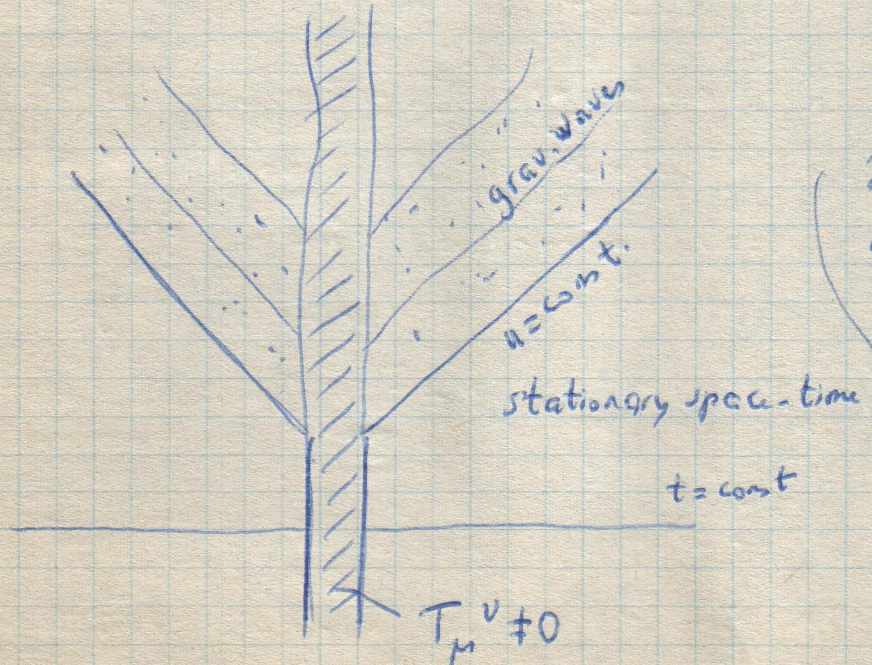
which follows from the Riemannian character of the geometry. $t_\mu{}^\nu$ ~~may be~~ ~~easily~~ is seen to be quadratic in the Γ's.

By integrating both sides of the differential conservation law (14) over a three-dimensional region V of X and using Stokes' theorem, we obtain

$$\int_V (T_\mu + t_\mu) = \int_S U_\mu$$

where $S = \partial V$ is the surface constituting the boundary of V.

Consider now a bounded gravitating system, which is initially stationary ~~and from $t = -\infty$ to some instant~~ at some instant of time but starts radiating gravitational waves. A somewhat naive picture of the corresponding space-time geometry is represented in the figure.



u denotes a null co-ordinate (a scalar function such that $g^{\mu\nu}\frac{\partial u}{\partial x^\mu}\frac{\partial u}{\partial x^\nu} = 0$) which is expected to generalize the retarded time $t - r$ of special relativity.