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Lectures on General Relativity [1]

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LECTURE I

BOUNDARY CONDITIONS IN GRAVITATIONAL RADIATION THEORY

1. In dealing with physical problems, we are often interested in the solution of field equations with given sources, but with nothing known about initial conditions. Therefore, we cannot solve the Cauchy problem, for although it is a very natural problem for hyperbolic normal equations, its solution requires a detailed knowledge of the field on an initial space-like hypersurface. However, in general, a whole set of fields corresponds to a given distribution of sources, and in order to find a unique solution of the physical problem we must specify some additional conditions. For linear field equations these conditions may consist in prescribing the form of the Green's function (e.g., retarded, advanced, etc.). If we investigate the field in the whole (unbounded) space-time we can ensure uniqueness by specifying some appropriate *boundary conditions* at spatial infinity. This latter approach has the advantage of being applicable to nonlinear equations such as Einstein's gravitational equations. These boundary conditions, first formulated for a periodic scalar field by Sommerfeld [1], have a definite physical meaning. For example, the "Ausstrahlungsbedingung" of Sommerfeld means that the system can lose energy in the form of radiation, but that no radiation is falling on the

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system from the exterior. We propose now to reformulate Sommerfeld's radiation condition so as to exhibit its physical meaning [2] and to generalize this condition to the case of Einstein's theory of gravitation [3].

2. Let us first take the *scalar wave equation*

$$\Delta\phi - \phi_{,00} = -4\pi f \tag{1}$$

and assume $f(\mathbf{r}, t)$ to be a regular function vanishing outside a bounded 3-dimensional region V. The retarded solution of (1) can be written in the form

$$\phi(\mathbf{r}, t) = \int_V \frac{f(\mathbf{r}', t - R)}{R} dV', \qquad R = |\mathbf{r} - \mathbf{r}'|. \tag{2}$$

From the formula (2) we obtain the following asymptotic values of ϕ and its derivatives [2]

$$\begin{aligned} \phi &= r^{-1} \int_V f(\mathbf{r}', t - R) dV' + O(r^{-2}), \\ \phi_{,\alpha} &= k_\alpha r^{-1} \int_V f_{,0}(\mathbf{r}', t - R) dV' + O(r^{-2}), \end{aligned} \tag{3}$$

where

$$k^\alpha = (1, n^s), \qquad n^s = x^s / r \tag{4}$$

is a null vector field.

Now, we can formulate the following boundary conditions to be imposed on solutions of (1):

$$\phi = O(r^{-1}), \tag{5}$$

there exists a function $\psi = O(r^{-1})$ *such that*

$$\phi_{,\alpha} = \psi k_\alpha + O(r^{-2}), \tag{6}$$

where k_α *is given by (4).*

We see from equations (3) that every retarded solution of (1) fulfills (5) and (6). Conversely, if the condition (6) is fulfilled, then ϕ satisfies Sommerfeld's radiation condition

$$\lim_{r\to\infty} r k^\alpha \phi_{,\alpha} = \lim_{r\to\infty} r\,(\partial\phi/\partial t + \partial\phi/\partial r) = 0.$$

[2] $F_A = O(r^{-k})$ means that there exists a constant M such that, for sufficiently large r, we have $|F_A| < Mr^{-k}$; Greek indices run from 0 to 3, Latin, from 1 to 3; $x^0 = t$, $(x^1, x^2, x^3) = \mathbf{r}$; a comma followed by an index denotes partial differentiation. The summation convention will be used throughout. Indices will be raised by means of the Galilean metric tensor $\eta^{\alpha\beta}$ ($\eta^{00} = 1$, $\eta^{ik} = -\delta^{ik}$, $\eta^{i0} = 0$). Square brackets stand for alternation, e.g., $F_{[\mu\lambda\alpha]} = F_{\mu\lambda\alpha} + F_{\lambda\alpha\mu} + F_{\alpha\mu\lambda} - F_{\alpha\lambda\mu} - F_{\lambda\mu\alpha} - F_{\mu\alpha\lambda}$.

Thus the wave equation with a spatially bounded source has always one and only one solution fulfilling our conditions (5) and (6). This is the retarded solution.

If we replace (4) by $k^\alpha = (1, -n^s)$, we obtain the conditions which characterize advanced solutions of (1).

Let us introduce the energy-momentum tensor of the field ϕ:

$$T_\mu{}^\lambda = L\delta_\mu{}^\lambda - \phi_{,\mu}\partial L/\partial\phi_{,\lambda}, \qquad \text{where} \quad L = -\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}/8\pi.$$

From the asymptotic expression for $\phi_{,\alpha}$ we have $L = O(r^{-3})$ and

$$4\pi T_{\mu\lambda} = \psi^2 k_\mu k_\lambda + O(r^{-3}). \tag{7}$$

Thus the asymptotic form of $T_{\mu\lambda}$ resembles the energy-momentum tensor of a perfect fluid with vanishing rest mass. We can obtain from (7) the time rate of radiation of energy and momentum:

$$4\pi W_\mu = 4\pi \oint_S T_\mu{}^s n^s dS = \oint_S \psi^2 k_\mu dS.$$

The integrals are to be taken over the surface of a sphere "at infinity." The condition (6) ensures that $W_0 \geqslant 0$.

3. The situation is somewhat more complicated in *electrodynamics* because of the gauge-invariance. Maxwell's equations

$$f^{\mu\lambda}{}_{,\lambda} = -4\pi j^\mu, \qquad f_{\mu\lambda} = A_{\lambda,\mu} - A_{\mu,\lambda} \tag{8}$$

can be reduced to four wave equations

$$\Delta A^\mu - A^\mu{}_{,00} = -4\pi j^\mu \tag{9}$$

if one imposes on the potentials A^μ the Lorentz condition

$$A^\alpha{}_{,\alpha} = 0. \tag{10}$$

We can impose conditions like (5) and (6) on A^α satisfying (9) and (10). It would perhaps be more satisfactory if we formulated the boundary conditions in a way involving only the *field* $f_{\mu\lambda}$ and not the potentials A^α. However, the chosen conditions will be more suitable for a straightforward generalization to the gravitational case. The current j^μ will satisfy the same regularity and boundedness conditions as did f in the scalar field case.

We formulate the boundary conditions as follows: *there exists a potential* A^μ *satisfying*

$$A^\mu = O(r^{-1}) \tag{11}$$

and four functions

$$B^\mu = O(r^{-1}) \tag{12}$$

such that $A_{\alpha,\beta} = B_\alpha k_\beta + O(r^{-2})$, *and*

$$B_\alpha k^\alpha = O(r^{-2}). \tag{13}$$

It should be noted that there are many sets of functions A^α which satisfy Maxwell's equations (8) and the conditions (11)–(13), but they differ only by gauge transformations and all represent the same electromagnetic field.

From equations (12) and (13) we obtain the asymptotic form of the field[3]:

$$f_{\mu\lambda} \cong k_\mu B_\lambda - k_\lambda B_\mu, \qquad k_\alpha B^\alpha \cong 0, \tag{14}$$

or, in vector notation:

$$\mathbf{E} \cong (\mathbf{B} \times \mathbf{n}) \times \mathbf{n}, \qquad \mathbf{H} \cong \mathbf{B} \times \mathbf{n}, \qquad B^\alpha = (B^0, \mathbf{B}), \qquad k^\alpha = (1, \mathbf{n}).$$

Equations (14) represent a system of "gauge-invariant" boundary conditions. The electromagnetic field has asymptotically the form of a plane wave. For the energy-momentum tensor

$$4\pi T_{\mu\lambda} = \tfrac{1}{4}\eta_{\mu\lambda} f_{\alpha\beta} f^{\alpha\beta} - f_{\mu\alpha} f_\lambda{}^\alpha$$

we obtain the expression

$$4\pi T_{\mu\lambda} = -B_\alpha B^\alpha k_\mu k_\lambda + O(r^{-3}).$$

Since k_ν is a null vector, it follows from (13) that $B_\alpha B^\alpha \leqslant 0$ for sufficiently large r.

The total charge e contained in the field can be calculated by means of the Gauss law

$$4\pi e = \oint f^{k0} n^k dS.$$

Though f^{k0} contains terms going as $1/r$, nevertheless e is finite by virtue of (14).

4. In physics we are ordinarily interested in conservation laws which have an integral character. A classical conserved quantity is a functional $f[\sigma]$ depending on a space-like hypersurface σ. A conservation law is the statement that, by virtue of the equations of motion, f does not in fact depend on σ. As is known, in general relativity the energy-momentum tensor of matter $T_{\mu\lambda}$ does not by itself lead to an integral conservation law. However, if we introduce an energy-momentum pseudotensor of the gravitational field $\underline{t}_\mu{}^\lambda = (\delta_\mu{}^\lambda \underline{G} + g^{\alpha\beta}{}_{,\mu} \partial \underline{G}/\partial g^{\alpha\beta}{}_{,\lambda})/2\kappa$, then the sum $\underline{T}_\mu{}^\lambda + \underline{t}_\mu{}^\lambda$ is divergence-free by virtue of Einstein's equations (17)[4].

[3] We shall sometimes write $F \cong G$ to mean $F = G + O(r^{-2})$.

[4] $g_{\mu\lambda}$ will denote the metric tensor of the Riemannian space-time V_4. Underlined letters denote tensor densities with respect to affine coordinate transformations; $\underline{G} = \sqrt{-g} g^{\mu\lambda} \left(\Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\lambda\alpha} - \Gamma^\alpha{}_{\mu\lambda}\Gamma^\beta{}_{\alpha\beta}\right)$.

The Einstein tensor density $\underline{G}_\mu{}^\lambda = \sqrt{-g}(R_\mu{}^\lambda - \frac{1}{2}\delta_\mu{}^\lambda R)$ can be written in the form

$$\underline{G}_\mu{}^\lambda = \kappa \left(\underline{t}_\mu{}^\lambda + \underline{U}_\mu{}^{\alpha\lambda}{}_{,\alpha} \right), \tag{15}$$

where the "superpotentials" $\underline{U}_\mu{}^{\alpha\lambda}$ are given by [4]

$$2\kappa \underline{U}_\mu{}^{\nu\lambda} = \sqrt{-g} g^{\beta[\alpha} \delta_\mu{}^\nu g^{\lambda]\tau} g_{\alpha\beta,\tau} = -2\kappa \underline{U}_\mu{}^{\lambda\nu}. \tag{16}$$

If Einstein's equations

$$G_{\mu\lambda} = -\kappa T_{\mu\lambda} \tag{17}$$

are satisfied, then equations (15) and (16) imply

$$\underline{T}_\mu{}^\lambda + \underline{t}_\mu{}^\lambda = \underline{U}_\mu{}^{\lambda\alpha}{}_{,\alpha}, \qquad \text{thus} \ \ (\underline{T}_\mu{}^\lambda + \underline{t}_\mu{}^\lambda)_{,\lambda} = 0. \tag{18}$$

The functions $\underline{t}_\mu{}^\lambda$ are not components of a tensor density (essentially because of the equivalence principle) and many physicists (e.g., Schrödinger [5], Bauer [6]) have raised doubts as to their physical meaning. Einstein [7] and F. Klein [8] formulated some conditions which enable us to consider the integrals

$$P_\mu[\sigma] = \int_\sigma \left(\underline{T}_\mu{}^\alpha + \underline{t}_\mu{}^\alpha \right) dS_\alpha = \oint_S \underline{U}_\mu{}^{\alpha\beta} dS_{\alpha\beta} \tag{19}$$

as representing the total energy and momentum of the system matter plus gravitational field. These conditions can be summarized as follows: Let us take an isolated system of masses ($T_\mu{}^\lambda = 0$ outside a bounded 3-region) and *assume* the existence of coordinates such that [9]

$$g_{\mu\lambda} = \eta_{\mu\lambda} + O(r^{-1}), \qquad g_{\mu\lambda,\alpha} = O(r^{-2}), \tag{20}$$

where r denotes the distance measured along geodesics from a fixed point on a space-like σ. Equations (20) have a double meaning: they constitute a system of boundary conditions and they distinguish a set of co-ordinate systems ("Galilean at infinity").

Using (18) it can be easily proved that:

1) $P_\mu[\sigma]$ calculated from (19) in a co-ordinate system satisfying (20) is always finite and does not depend on σ;

2) P_μ is unaltered by a co-ordinate change which preserves (20) and reduces to the identity for $r \to \infty$;

3) P_μ is a vector with respect to linear orthogonal transformations. The proof is based on the vanishing of the integrals

$$p_\mu = \int_\Sigma (\underline{T}_\mu{}^\lambda + \underline{t}_\mu{}^\lambda) dS_\lambda \tag{21}$$

taken over a *time-like* "cylindrical" hypersurface Σ at spatial infinity (note that S appearing in (19) is the intersection of Σ and σ). The vanishing of these integrals is ensured by (20) ($\underline{t}_\mu{}^\lambda$ is quadratic in $g_{\mu\lambda,\alpha}$) and our assumption on $T_{\mu\lambda}$ (Fig. 1).

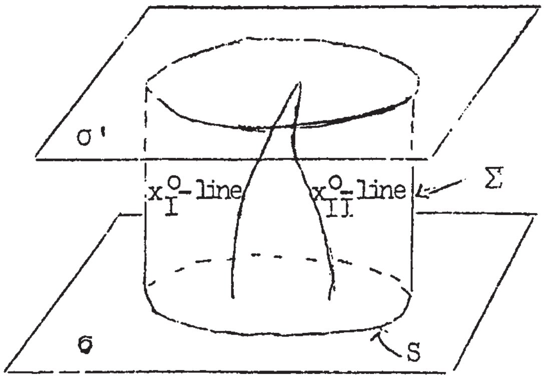

Figure 1.

The proof of part 2) of the Einstein-Klein theorem is as follows. Let us take two space-like hypersurfaces σ and σ' and choose on σ two coordinate systems ${x_I}^\alpha$ and ${x_{II}}^\alpha$ which coincide at infinity and satisfy the conditions (20). Now, introduce two coordinate systems in the whole space-time which are identical on σ' and coincide respectively with ${x_I}^\alpha$ and with ${x_{II}}^\alpha$ on σ. Between σ and σ' these coordinates are supposed to satisfy (20). Applying the Gauss theorem twice to (18) in the region lying between σ and σ', and taking into account $p^I_\mu = 0 = p^{II}_\mu$ we obtain $P^I_\mu[\sigma] = P_\mu[\sigma']$ and $P^{II}_\mu[\sigma] = P_\mu[\sigma']$, and thus $P^I_\mu[\sigma] = P^{II}_\mu[\sigma]$. The integrals (21) can eventually be identified with the total energy and momentum radiated through Σ, and Lichnerowicz's boundary conditions (20) automatically exclude the existence of any radiation.

5. Comparison with electrodynamics suggests that radiation fields in general relativity should be characterized by $g_{\mu\lambda,\alpha} \sim 1/r$, rather than by $g_{\mu\lambda,\alpha} \sim 1/r^2$. However, if the integrals (21) do not vanish, the proof of the Einstein-Klein theorem is no longer valid and fresh doubts as to the meaning of (19) arise. We propose to generalize the boundary conditions (20) in such a way as to include radiation fields. We expect that these conditions will ensure the finiteness of P_μ and that P_μ will not change with coordinate transformations which reduce to an identity for $r \to \infty$ and preserve the *form* of the boundary conditions. The dependence of P_μ on σ will now correspond to the loss of total energy by radiation.

Fock [6] proposes to normalize the coordinate systems by means of de Donder's relation

$$\underline{g}^{\mu\lambda}{}_{,\lambda} = 0 \tag{22}$$

and imposes on $g_{\mu\lambda}$ the radiation condition of Sommerfeld. We find this formulation somewhat stringent. In particular, we see no reasons for restricting ourselves to harmonic coordinates only. There is no convincing argument for writing the Schwarzschild line element in harmonic coordinates instead of, say, in the isotropic ones. The possibility of introducing privileged coordinates in Riemannian space-time is known to be closely related to its symmetry properties [10]. Galilean coordinates and Lorentz transformations reflect the homogeneity and the isotropic properties of flat space-time [11]. The flat metric tensor is *invariant in form* with respect to Lorentz transformations, which constitute a 10-parameter group of motions of space-time. Fock's "Lorentz transformations" (= linear orthogonal transformations in curvilinear, harmonic coordinates) have not this property. However, if a space-time is flat at infinity, it seems reasonable to distinguish a set of coordinates which exhibit this "asymptotical symmetry" and this is the meaning of conditions (20).

We generalize the conditions of Fock along the lines presented in the preceding sections. First, introduce a null vector field k^α defined as follows. Let n^α be a unit space-like vector lying in σ, perpendicular to the "sphere" $r =$ const., and pointing outwards. We put $k^\alpha = n^\alpha + t^\alpha$, where t^α denotes a unit time-like vector normal to σ, such that $t^0 > 0$.

Now, we formulate the following boundary conditions to be imposed on gravitational fields due to isolated systems of matter: *there exist coordinate systems and functions $i_{\mu\lambda} = O(r^{-1})$ such that*

$$g_{\mu\lambda} = \eta_{\mu\lambda} + O(r^{-1}), \qquad g_{\mu\lambda,\alpha} = i_{\mu\lambda} k_\alpha + O(r^{-2}), \tag{23}$$

$$(i_{\mu\lambda} - \tfrac{1}{2}\eta_{\mu\lambda}\eta^{\alpha\beta} i_{\alpha\beta})k^\lambda = O(r^{-2}). \tag{24}$$

These conditions correspond to the "Ausstrahlungsbedingung" of Sommerfeld; we obtain the "Einstrahlungsbedingung" if we assume n^α to be inward-pointing instead of outward-pointing. Relations (23) and (24) together are *weaker* than (20); this means that every field fulfilling (20) satisfies also conditions (23) and (24) and that the class of coordinate systems distinguished by (23) and (24) is larger than that defined by (20). Equation (24) restricts the coordinate systems to those which are asymptotically harmonic; however, it may be noted that, for example, the isotropic coordinates used in Schwarzschild space-time are asymptotically harmonic in this sense.

Strictly speaking, the justification of conditions (23) and (24) should await the proof that Einstein's equations with bounded sources have always exactly one solution satisfying them.

6. We shall now present some consequences of (23) and (24). First of all, we shall examine the convergence of the energy integrals (19). The superpotentials are linear in $g_{\mu\lambda,\alpha}$ and thus go as $1/r$; we must therefore show that the terms behaving as $1/r$ cancel out in the surface integral (19). Indeed, the surface element $dS_{\lambda\mu}$ is proportional to $n_{[\lambda}t_{\mu]} = n_{[\lambda}k_{\mu]}$, and the terms in question in (19) can be written as $\eta^{\beta[\alpha}\delta_\mu{}^\nu\eta^{\lambda]\tau} i_{\alpha\beta}k_\tau k_{[\nu}n_{\lambda]}$. Taking into account (24) we verify that this expression does vanish.

Let us take a coordinate transformation

$$x^\alpha \to x'^\alpha = x^\alpha + a^\alpha(x) \tag{25}$$

fulfilling the conditions

$$a^\alpha = O(r), \qquad a_{\alpha,\beta} = b_\alpha k_\beta + O(r^{-2}), \tag{26}$$

where $a_\alpha = \eta_{\alpha\beta}a^\beta$, $b_\alpha = O(r^{-1})$, and

$$a_{\alpha,\mu\lambda} = b_{\alpha,\mu}k_\lambda + O(r^{-2}), \qquad b_{\alpha,\lambda} = O(r^{-1}). \tag{27}$$

From equation (27) follows the existence of functions $c_\mu = O(r^{-1})$ such that

$$b_{\lambda,\mu} = c_\lambda k_\mu + O(r^{-2}). \tag{28}$$

Coordinate transformations (25) satisfying (26) and (27) preserve the *form* of the boundary conditions; this can easily be seen from the transformation formulae for $g_{\mu\lambda}$ and $i_{\mu\lambda}$:

$$g'_{\mu\lambda}(x') \cong g_{\mu\lambda}(x) + b_\mu k_\lambda + b_\lambda k_\mu,$$

$$i'_{\mu\lambda}(x') \cong i_{\mu\lambda}(x) + c_\mu k_\lambda + c_\lambda k_\mu. \tag{29}$$

Computing the superpotentials in both coordinate systems and taking into account the relations (23)–(29) we obtain

$$\underline{U'}_\mu{}^{\alpha\lambda} k'_{[\alpha} n'_{\lambda]} = \underline{U}_\mu{}^{\alpha\lambda} k_{[\alpha} n_{\lambda]} + O(r^{-3}).$$

Therefore the total energy and momentum P_μ is well defined by equation (19) and the boundary conditions (23), (24). It must be noted that our prescription demands that the *calculation* of P_μ should be performed by means of (19) using coordinates which satisfy equations (23) and (24). This does not by any means imply that the energy is only a property of the coordinate system. The vector $P_\mu[\sigma]$ constitutes a *global* characteristic of the *field* and it is only for computational purposes that we must appeal to (23), (24).

7. The total energy and momentum p_μ radiated between two hypersurfaces σ and σ' is given by (21), or by

$$p_\mu = P_\mu[\sigma] - P_\mu[\sigma'] = \int_\Sigma \underline{t}_\mu{}^\lambda dS_\lambda$$

($T_{\mu\lambda}$ vanishes on Σ). The boundary conditions enable the estimation of p_μ; we have in fact

$$\underline{t}_\mu{}^\lambda = \alpha k_\mu k^\lambda + O(r^{-3}) \tag{30}$$

where

$$4\kappa\alpha = i^{\mu\lambda}(i_{\mu\lambda} - \tfrac{1}{2}\eta_{\mu\lambda}\eta^{\alpha\beta}i_{\alpha\beta}). \tag{31}$$

α is "$\cong$ invariant" with respect to the transformation (29) and is *non-negative* by virtue of (24); therefore $p_0 \geqslant 0$. The existence of radiation is characterized by $p_\mu \neq 0$.

We could also take a more general case, including the electromagnetic field. The boundary conditions for $g_{\mu\lambda}$ should be supplemented by those for $f_{\alpha\beta}$ given by (14). We obtain in this case $\underline{T}_\mu{}^\lambda + \underline{t}_\mu{}^\lambda = \bar{\alpha}k_\mu k^\lambda + O(r^{-3})$, $0 \leqslant \bar{\alpha} = O(r^{-2})$.

8. Pirani [12] and Lichnerowicz [13] proposed recently definitions of *pure* radiation fields. It may be interesting to compare their definitions with our approach. Let us admit the additional but reasonable assumption that the second derivatives of $g_{\mu\lambda}$ also go to 0 as $1/r$ and that $g_{\mu\lambda,\alpha\beta} \cong i_{\mu\lambda,\beta}k_\alpha$. From $i_{\mu\lambda,\alpha}k_\beta \cong i_{\mu\lambda,\beta}k_\alpha$ follows the existence of functions $j_{\mu\lambda} = O(r^{-1})$ such that

$$g_{\mu\lambda,\alpha\beta} \cong j_{\mu\lambda}k_\alpha k_\beta, \qquad (j_{\mu\lambda} - \tfrac{1}{2}\eta_{\mu\lambda}\eta^{\alpha\beta}j_{\alpha\beta})k^\lambda \cong 0. \tag{32}$$

For the curvature tensor we get

$$R_{\mu\lambda\alpha\beta} \cong \tfrac{1}{2}k_{[\mu}j_{\lambda][\alpha}k_{\beta]} \tag{33}$$

The principal part of $R_{\mu\lambda\alpha\beta}$ has therefore the same form as a discontinuity of the Riemann tensor [14] and is thus of type II, with vanishing scalar invariants, in the Petrov-Pirani classification [12]. It is interesting to note that the plane gravitational waves discovered by Bondi and Robinson [15], [38] are also of type II "pure radiation." It seems that in the theory of gravitation we have essentially the same situation as in electrodynamics: a gravitational wave produced by a system of bodies behaves at large distances locally as a plane wave. L. Marder pointed out that the Riemann tensor of outgoing cylindrical waves [16]–[18] goes for large r like $r^{-1/2}$ (r denotes here the "radial" coordinate) and is asymptotically of type II. This result seems to confirm the general theory; the behaviour like $r^{-1/2}$ is to be expected for fields with cylindrical symmetry.

The terms proportional to $1/r$ in $R_{\mu\lambda}$ cancel out by virtue of (24). Conversely, $R_{\mu\lambda} \cong 0$ and equation (18) imply $R_{\mu\lambda\alpha\beta} \cong 0$ unless $k_\mu k^\mu = 0$. If we take into account the electromagnetic field, Einstein's equations can be written in the form

$$R_{\mu\lambda} = \beta k_\mu k_\lambda + O(r^{-3}), \qquad \beta = O(r^{-2}). \tag{34}$$

Moreover, it follows from (33) that

$$k_{[\mu}R_{\lambda\alpha]\beta\gamma} \cong 0, \qquad k^\mu R_{\mu\lambda\alpha\beta} \cong 0. \tag{35}$$

If one replaces the asymptotic equalities $\cong$ by strict ones, then equations (34) and (35) become Lichnerowicz's conditions [13] characterizing a pure radiation field.

Our boundary conditions contain not only the characterization of the field but also some conditions on the coordinates. It would be very interesting to formulate purely geometrical boundary conditions (e.g. in terms of scalar invariants of the curvature tensor). But the principal unsolved problem is rather whether there are any non-stationary gravitational fields produced by bounded systems of matter and flat at spatial infinity. The theory presented here has of course been developed on the assumption that such fields exist.

LECTURES II & III

EQUATIONS OF MOTION AND GRAVITATIONAL RADIATION

The practical applications of electromagnetic radiation theory are connected with the possibility of producing waves with arbitrary time-dependence. Maxwell's equations impose no conditions on the motion of charges; by means of nonelectrical forces we can move them in a quite arbitrary way. The situation is different in General Relativity: here the field equations restrict the motions of masses, and the question arises whether or not these restrictions may prevent gravitational radiation from taking place.

The connection between Einstein's field equations and the equations of motion has been known for a long time and is quite elementary; for example, if we write the field equations for a perfect fluid without pressures

$$G^{\mu\lambda} = -\kappa T^{\mu\lambda} = -\kappa\rho u^{\mu} u^{\lambda}, \tag{1}$$

then, from the Bianchi identities we have $G^{\mu\lambda}{}_{;\lambda} \equiv 0$ which implies $T^{\mu\lambda}{}_{;\lambda} = 0$, or

$$\left(\rho u^{\lambda}\right)_{;\lambda} = 0 \qquad (2a), \qquad \text{and} \qquad \rho Du^{\alpha}/ds = 0. \qquad (2b)$$

Equation (2a) expresses the law of conservation of mass and (2b) states that the trajectories of u^{ν} are geodesics (D denotes the absolute derivative). This idea can be generalized; let us take a classical field (not $g_{\mu\lambda}$) interacting with "pole-particles" and assume that the field equations are derivable from a Lorentz-invariant variational principle. Entirely from considerations of invariance (Noether's theorem), we obtain the following identity [10]

$$T^{\mu\lambda}{}_{;\lambda} + M^{\mu}(\text{eqs. of motion}) + N^{\mu}(\text{field eqs.}) \equiv 0,$$

where $T^{\mu\lambda}$ is the total energy-momentum tensor. M^{μ} and N^{μ} vanish if the equations of motion and the field equations are satisfied and $M^{\mu} = 0$ *implies* the equations of motion. In special relativity, we infer the conservation laws from $M^{\mu} = 0 = N^{\mu}$. In the theory of gravitation, where $T^{\mu\lambda}$ acts as a source of the g-field, $T^{\mu\lambda}{}_{;\lambda} = 0$ *must* hold (because of the Bianchi identities) and if $N^{\mu} = 0$ then also $M^{\mu} = 0$. It is not necessary to postulate separately a dynamical principle for the motion of particles in general relativity.

As is known, Einstein regarded the energy tensor as a temporary means for the description of matter and sought for a description of nature in terms of purely "geometrical" fields. One of the provisional solutions was to treat particles as singularities in empty space-time. The main purpose of the famous paper by Einstein, Infeld and Hoffmann [19] was to show that the motion of *singularities* is also determined by the field equations and to work out an approximation method suited to the calculations of relativistic corrections to the Newtonian motion of celestial bodies. The equations of motion were obtained from the vanishing of some surface integrals surrounding the singularities which expressed the integrability conditions for the approximate field equations. The original method of EIH was improved in a later paper by Einstein and Infeld [20], by the introduction of some pole and dipole terms in such a way that the integrability conditions were satisfied automatically. The equations of motion were then obtained by setting equal to zero the sums of these pole and dipole moments.

The problem of motion was attacked also by Fock [21], [22], [11] and his students [23], [24]. They used the same approximation method as Einstein and Infeld did, but the bodies were represented not by singularities but by a continuous energy-momentum tensor with pressures. Fock fixed the space-time coordinate system by the de Donder condition and obtained the equations of motions of the centre of inertia of a body by integrating the equations $gT^{\mu\lambda}{}_{;\lambda} = 0$ over the 3-region occupied by it. He obtained also some equations for the internal motion of rotating bodies (from the equations $\int x^{[i}T^{k]\alpha}{}_{;\alpha} g dV = 0$).

Infeld [25] introduced an energy-momentum tensor involving Dirac δ-functions for the description of pole particles. This produced a great simplification in the derivation of the post-Newtonian equations of motion (obtained from $T^{\mu\lambda}{}_{;\lambda} = 0$).

Einstein, Infeld and Hoffmann had assumed certain forms of series expansion of the metric tensor which by analogy with electrodynamics they interpreted as corresponding to the choice of the symmetric (half-advanced, half-retarded) Green's function. Infeld [26] wrote down the first terms in $g_{\mu\lambda}$ corresponding to the choice of a retarded Green's function and showed that they did not give any contribution to the equations of motion up to the 7th order (the Newtonian equations are of the 4th order and the post-Newtonian ones – found by EIH – of the 6th order). N. Hu [27]worked out the radiation terms in the next step and found "anti-damping" – the energy of a system of two bodies appeared to *increase* when the radiation was taken into account. The first radiation terms are functions of the time alone and several papers dealt with the problem whether they represent a "true" gravitational field or could be "annihilated" by a coordinate transformation [28]–[3l]. An answer to this question will be proposed below.

The extent to which the equations of motion do depend on the choice of coordinates is a problem which has drawn some attention in recent years [32]–

[34]. We shall briefly discuss this and some other aspects of the EIH method, mainly those related to the problem of gravitational radiation.

1. *The "new approximation method*." Let us again start with the scalar wave equation

$$\Box\phi = \Delta\phi - \phi_{,00} = 0, \tag{3}$$

and introduce the time t instead of the "cotime" $x^0 = ct$. If a solution $\phi(x^0, x^k, c) = \phi(ct, x^k, c)$ of (3) can be expanded into a power series in $1/c$

$$\phi = \sum_{n=0}^{\infty} c^{-n} \underset{n}{\phi}(t, x^k), \tag{4}$$

then the functions $\underset{n}{\phi}$ satisfy

$$\Delta \underset{0}{\phi} = 0, \qquad \Delta \underset{1}{\phi} = 0, \qquad \Delta \underset{2}{\phi} = \underset{0}{\ddot{\phi}}, \ldots, \Delta \underset{k}{\phi} = \underset{k-2}{\ddot{\phi}}, \ldots \tag{5}$$

(the dots over the ϕ's stand for derivatives with respect to t). The structure of (5) is such that we *can*, if we wish, find solutions (4) containing only even or only odd terms. If we put $\underset{0}{\phi} = 0$, $\underset{2n-1}{\phi} = 0$ $(n = 1, 2, \ldots)$, start with the pole solution in the second order; $\underset{2}{\phi} = a(t)/r$, and take the simple solutions $\underset{4}{\phi} = \frac{1}{2}\ddot{a}r$, $\underset{6}{\phi} = (4!)^{-1} r^3 d^4a/dt^4, \ldots$, then we obtain the standing wave solution of (3):

$$2\phi = a(t - r/c) + a(t + r/c).$$

A retarded solution can be obtained if we introduce a "first radiation term" in the 3rd order:

$$\underset{0}{\phi} = 0, \qquad \underset{1}{\phi} = 0, \qquad \underset{2}{\phi} = a/r, \qquad \underset{3}{\phi} = -\dot{a}, \qquad \underset{4}{\phi} = \tfrac{1}{2}\ddot{a}r,$$

$$\underset{5}{\phi} = (3!)^{-1}\,\dddot{a}r^2, \ldots; \phi = a(t - r/c)/r.$$

It is important to note that $\underset{n}{\phi} = O(r^{n-3})$ for $r \to \infty$ and this is also a general property of solutions of the inhomogeneous wave equation with a spatially bounded source. If λ is the characteristic wavelength of the field, then we can safely stop after the few first terms of the series (4) only in the region where

$$r \ll \lambda. \tag{6}$$

In other words the new approximation method of EIH is not well suited for the description of a field in the wave zone. If we write Maxwell's equations in the form

$$\Box A^{\alpha} = -4\pi j^{\alpha}, \qquad A^{\alpha}{}_{,\alpha} = 0, \qquad j^{\alpha}{}_{,\alpha} = 0 \tag{7}$$

and assume that j^0 is of order 2 and j^k of order 3, then the *retarded* solution of (7) can be expanded into a power series as follows (in future we shall put $c = 1$):

$$A^0(\mathbf{r}, t) = \int \underset{2}{\overset{0}{j}}(\mathbf{r}', t)/R\,dV' - \int \underset{2}{\overset{0}{j}}{}_{,0}\, dV' + (2!)^{-1} \int \underset{2}{\overset{0}{j}}{}_{,00}\, R\,dV' + \ldots$$

$$A^k(\mathbf{r}, t) = \int \underset{3}{\overset{k}{j}}(\mathbf{r}', t)/R\,dV' - \int \underset{3}{\overset{k}{j}}{}_{,0} dV' + (2!)^{-1} \int \underset{3}{\overset{k}{j}}{}_{,00} R\,dV' + \ldots.$$

The conservation of charge implies $\underset{3}{A}{}^0 = 0$ and the first radiation term appears only in the 4th order ($\underset{4}{A}{}^k$). For large values of r and for $n \geqslant 3$ we have $\underset{n}{A}{}^{\alpha} = O(r^{n-4})$.

In the linearized theory of gravitation the situation is similar but the radiation terms are shifted still further along the series. If we write $\underline{g}^{\mu\lambda} = \sqrt{-g}g^{\mu\lambda} = \eta^{\mu\lambda} - \gamma^{\mu\lambda}$ and assume de Donder's conditions $\gamma^{\mu\lambda}{}_{,\lambda} = 0$, then the linearized Einstein's equations become

$$\Box\gamma^{\mu\lambda} = +16\pi T^{\mu\lambda}, \qquad T^{\mu\lambda}{}_{,\lambda} = 0. \tag{8}$$

T^{00} can be assumed to be of order 2, T^{0k} of order 3 and T^{kl} of order 4. This corresponds to the EIH assumption that the mass is of 2nd order. Expanding into a power series the retarded solution

$$\gamma^{\mu\lambda}(\mathbf{r}, t) = -4 \int dV' T^{\mu\lambda}(\mathbf{r}', t - R)/R \tag{9}$$

of equation (8), we find that

$$\underset{2}{T}{}^{00}{}_{,0} + \underset{3}{T}{}^{0k}{}_{,k} = 0 \quad \text{implies} \quad \underset{3}{\gamma}{}^{00} = 0, \quad \text{and}$$

$$\underset{3}{T}{}^{k0}{}_{,0} + \underset{4}{T}{}^{kl}{}_{,l} = 0 \quad \text{implies} \quad \underset{4}{\gamma}{}^{0k} = 0. \tag{10}$$

Thus $\underset{5}{\gamma}{}^{ik}$ is the first non-vanishing radiation term, and from (9) and (10):

$$\underset{n}{\gamma}{}^{\mu\lambda} = O(r^{n-5}) \quad \text{for} \quad n \geqslant 4. \tag{11}$$

In the theory of gravitation we have

$$g_{\mu\lambda} = \sum_{n=0}^{\infty} \underset{n}{g}{}_{\mu\lambda} \tag{12}$$

where $\underset{0}{g}_{\mu\lambda} = \eta_{\mu\lambda}$ and $\underset{1}{g}_{\mu\lambda} = 0$. Expanding $R_{\mu\lambda}$ into a power series we obtain equations for $\underset{n}{g}_{\mu\lambda}$ which, in empty space-time, have the form

$$0 = \underset{n}{R}_{\mu\lambda} = \text{linear function of } \underset{n}{g}_{\mu\lambda,ik},\quad \underset{n-1}{\dot{g}}_{\mu\lambda,i},\quad \underset{n-2}{\ddot{g}}_{\mu\lambda}$$

$$+ \text{nonlinear function of } \underset{n-2}{g}_{\mu\lambda}, \ldots, \underset{2}{g}_{\mu\lambda}.$$

Thus a solution for any $\underset{n}{g}_{\mu\lambda}$ will contain both terms of linear origin and terms of nonlinear origin. For example

$$\underset{4}{g}_{00} = \text{term coming from } \underset{2}{\ddot{g}} + \text{ terms coming from } \underset{2}{g} \cdot \underset{2}{g}.$$

The first terms give rise to the same limitation as in electrodynamics: $r \ll \lambda$. If we apply the EIH method to a system of bodies whose masses are of order m then the nonlinear terms in $\underset{4}{g}_{00}$ contain expressions like m^2/r^2 and we must have $r \gg m$. Further, if v is a characteristic velocity and l denotes a distance between the bodies we must have $r = l \ll \lambda$ or $v \ll 1$. In sum, the applicability of the EIH method is limited by the following conditions

$$m \ll r \ll \lambda, \qquad v \ll 1.$$

The first of these inequalities, which is connected with the nonlinearity of Einstein's equations, is common to this and other approximation methods. The second and third limitations are due to the distinguished role played by the time in the EIH method. It follows from these that the method is not well suited to the description of radiative phenomena.

The linear part of $\underset{n}{g}_{\mu\lambda}$ can easily be calculated from (9). We may expect $\underset{n}{g}_{\mu\lambda}$ also to go like r^{n-5} ($n \geqslant 4$), unless some nonlinear terms in $\underset{n}{g}_{\mu\lambda}$ cancel out the r^{n-5} terms in the linear part. In general, we cannot impose on the *expanded* metric the condition $\lim_{r\to\infty} \underset{n}{g}_{\mu\lambda} = 0$. However, this does not necessarily mean that the metric is non-flat at infinity.

2. Equations of motion. The equations of motion of singularities were obtained by Einstein, Infeld and Hoffmann [19] from the vanishing of certain surface integrals. The basic idea of this method can be explained in terms of electrodynamics; there the conservation of charge is an "equation of motion" which follows from the field equations alone. Assuming that A_α has been expanded into a power series, we can write Maxwell's equations in the form

$$\underset{n}{A}_{0,ss} = \underset{n-1}{A}_{s,s0}, \tag{13a}$$

$$\underset{n+1}{A}{}_{r,ss} - \underset{n+1}{A}{}_{s,rs} = \underset{n-1}{A}{}_{r,00} - \underset{n}{A}{}_{0,0r} \tag{13b}$$

If, as before, we put $\underset{0}{A}{}_{\alpha} = \underset{1}{A}{}_{\alpha} = 0$, then $\underset{2}{A}{}_{0}$ satisfies a Laplace equation and we may take $\underset{2}{A}{}_{0} = e(t)/r$ where $e(t)$ is an arbitrary function of time. Equations (13b), which in the present case become

$$\text{rot rot } \underset{3}{\mathbf{A}} = -\text{grad } \underset{2}{\dot{A}}{}_{0}, \quad \mathbf{A} = (A_1, A_2, A_3) \tag{13c}$$

are not independent; the divergence of the left hand side of (13c) vanishes identically (“strongly”). The divergence of the right-hand side also vanishes, by virtue of (13a). However, this is not sufficient to ensure the integrability of (13b) or (13c). The flux of rot rot $\underset{3}{\mathbf{A}}$ through a closed surface vanishes, and so also must the flux of grad $\underset{2}{\dot{A}}{}_{0}$. The equation $\Delta \underset{2}{A}{}_{0} = 0$ tells us that the flux of grad $\underset{2}{\dot{A}}{}_{0}$ does not depend on the shape of the surface (provided that we do not cross the singularity when deforming the surface). This means that the vanishing of the flux imposes a condition only on the singularity itself. We can calculate the flux of $-$grad $\underset{2}{\dot{A}}{}_{0}$ through a sphere $r = \text{const}$; this turns out to be $4\pi\dot{e}$. Therefore e must be a constant.

The situation is analogous in Einstein’s theory and can be presented in a concise form if one uses the superpotentials [4] (lecture I). The empty space field equations $\underline{G}_{\mu}{}^{k} = 0$ may be written

$$\underline{U}_{\mu}{}^{sk}{}_{,s} + \underline{U}_{\mu}{}^{0k}{}_{,0} + \underline{t}_{\mu}{}^{k} = 0. \tag{14}$$

Contracting with n_k and integrating over a closed surface we obtain (since $\underline{U}_{\mu}{}^{ks}$ is skew in k and s!)

$$\frac{d}{dt}\oint \underline{U}_{\mu}{}^{0k} n_k dS + \oint \underline{t}_{\mu}{}^{k} n_k dS = 0, \qquad \mu = 0, 1, 2, 3. \tag{15}$$

If we have an exact solution of the field equations, then (15) is identically satisfied and does not tell us anything. But if we use the EIH approximation method, and if we expand (14) then the conditions (15) written up to the l-th order will contain only known fields (of order $< l$) and will give non-trivial equations of motion (for $\mu = 1, 2, 3$). Equation (15) for $\mu = 0$ gives the conservation of energy.

Let us illustrate this by the simplest case, the Newtonian equations. From $\underset{2}{R}{}_{00} = 0$ we have

$$\Delta \underset{2}{g}{}_{00} = 0. \tag{16}$$

As a solution of this equation we may take

$$\underset{2}{g}{}_{00} = -\sum 2m/r, \tag{17}$$

where m denotes the mass of a body and r is distance from it; the summation is to be carried out over all particles. $\underset{2}{R}_{ik} = 0$ gives the equation for $\underset{2}{g}_{ik}$; it appears that a possible solution is

$$\underset{2}{g}_{ik} = \delta_{ik} \underset{2}{g}_{00}. \tag{18}$$

The lowest order fields are linear in the masses and therefore can also be evaluated from (9); g_{0k} is at least of order 3 and the problem of radiation does not appear before the 5th order. The knowledge of $\underset{2}{g}_{00}$ and $\underset{2}{g}_{ik}$ is sufficient for writing down the following surface integral

$$\frac{d}{dt} \oint \underset{2}{\underline{U}}_0{}^{k0} n_k dS = 0$$

($\underline{t}_0{}^k$ is of order 5 at least). Evaluating this integral around each of the singularities, we get

$$m = \text{const}.$$

The field equations for $\underset{3}{g}_{0k}$

$$\underset{3}{g}_{0k,ss} - \underset{3}{g}_{0s,ks} = \underset{2}{g}_{ks,0s} - \underset{2}{g}_{ss,0k}$$

are now integrable (since $m = \text{const.}$!) and lead to

$$\underset{3}{g}_{0k} = \sum 4m\dot{y}^k/r \tag{19}$$

where $y^k = y^k(t)$ are the coordinates of a particle, as yet arbitrary. The following surface integrals give the *Newtonian equations of motion*:

$$\frac{d}{dt} \oint \underset{3}{\underline{U}}_i{}^{k0} n_k dS + \oint \underset{4}{\underline{t}}_i{}^{k0} n_k dS = 0.$$

Infeld [25], [37] developed a formalism in which particles are treated as singularities described by means of δ functions. In this formalism it is necessary to define the value of some singular functions on the world lines of the particles. If $\phi(t, x^k, y^s(t))$ is a function depending on a world-line y^s and singular on this world-line (e.g. $\phi = |\mathbf{r} - \mathbf{y}(t)|^{-1}$) then

$$\tilde{\phi}(t) = (\phi - \text{part of } \phi \text{ singular at } x = y)_{x^k = y^k}.$$

For a regular function ϕ we can write

$$\tilde{\phi} = \int \phi(t, x^k, y^s)\delta_{(3)}\left(x^s - y^s\right) dV. \tag{20}$$

Infeld and Plebański [35] introduced some “good” δ functions which allow us to write an equation like this even for singular functions ϕ. The modified $\hat{\delta}(x)$ is defined by its regular model $\hat{\delta}(\alpha, x)$ which possesses the following properties:

$$\hat{\delta}(x)\text{"} = \text{"} \lim_{\alpha\to 0} \hat{\delta}(\alpha, x) = 0 \qquad \text{for } x \neq 0$$

$$\lim_{\alpha\to 0} \int_{-\infty}^{\infty} \hat{\delta}(\alpha, x) f(x) dx = f(0) \qquad \text{for a continuous } f$$

$$\lim_{\alpha\to 0} \int_{-\infty}^{\infty} \hat{\delta}(\alpha, x)|x|^{-k} dx = 0 \quad \text{for } k = 1, 2, \ldots p.$$

The $\sim$ operation is not distributive in general but we shall assume that it is so when applied to functions occurring in our work: $\widetilde{\alpha\beta} = \tilde{\alpha}\tilde{\beta}$. The energy-momentum tensor density of a system of pole particles can now be written

$$\underline{T}^{\alpha\beta} = \sum \int_{-\infty}^{\infty} \mu^{\alpha\beta} \hat{\delta}_{(4)} \left(x^\lambda - y^\lambda(\tilde{s})\right) d\tilde{s} = \sum \mu^{\alpha\beta} \hat{\delta}_{(3)} \left(x^k - y^k(t)\right) d\tilde{s}/dt \tag{21}$$

where $\tilde{s}$ is defined by $d\tilde{s}^2 = \widetilde{g_{\alpha\beta}} dy^\alpha dy^\beta$. It was shown by Tulczyjew [36] that $\mu^{\alpha\beta} = m_0 y'^\alpha y'^\beta$ $\left(y'^\alpha = dy^\alpha/d\tilde{s}\right)$ and $m_0 =$ const. We can rewrite (21) in the form

$$\underline{T}^{\alpha\beta} = \sum m\dot{y}^\alpha \dot{y}^\beta \hat{\delta}_{(3)}(\mathbf{x} - \mathbf{y}), \quad m = m_0 dt/d\tilde{s}, \quad \dot{y}^\alpha = dy^\alpha/dt.$$

The equations of motion are obtained by integrating $\underline{T}^{\alpha\beta}{}_{;\beta} \equiv \underline{T}^{\alpha\beta}{}_{,\beta} + \underline{T}^{\mu\lambda}\left\{{\alpha \atop \mu\lambda}\right\} = 0$ over the neighbourhood of one particle:

$$\begin{aligned} 0 = \int \underline{T}^{\alpha\beta}{}_{;\beta} dV &= \int \left[(m\dot{y}^\alpha \dot{y}^\beta \hat{\delta}_{(3)})_{,\beta} + m\dot{y}^\mu \dot{y}^\lambda \left\{ {\alpha \atop \mu\lambda} \right\} \hat{\delta}_{(3)} \right] dV \\ &= (m\dot{y}^\alpha)^{\cdot} + m \widetilde{\left\{ {\alpha \atop \mu\lambda} \right\}} \dot{y}^\mu \dot{y}^\lambda. \end{aligned}$$

It follows from this that $m d\tilde{s}/dt = m_0 =$ const and that

$$m_0 \left(\frac{d^2 y^\alpha}{d\tilde{s}^2} + \widetilde{\left\{ {\alpha \atop \mu\lambda} \right\}} \frac{dy^\mu}{d\tilde{s}} \frac{dy^\lambda}{d\tilde{s}} \right) = 0. \tag{22}$$

The equations of motion of heavy bodies have thus also the form of “geodesic” equations. We can eliminate ds from (22) and write the 3 equations of motion in the form

$$\ddot{y}^k + \left(\widetilde{\left\{ {k \atop \mu\lambda} \right\}} - \widetilde{\left\{ {0 \atop \mu\lambda} \right\}} \dot{y}^k \right) \dot{y}^\mu \dot{y}^\lambda = 0. \tag{23}$$

In this notation the Newtonian equations read $\ddot{y}^k + 2\widetilde{\left\{ {k_2 \atop 00} \right\}} = 0$. It can be easily seen from (23) that if we know the equations of motion of the n-th order, then we will be able to write $(n+1)$th order equations if we calculate $\underset{n-3}{g}{}_{ik}$, $\underset{n-2}{g}{}_{0k}$ and $\underset{n-1}{g}{}_{00}$. However, it has been shown by Bażański that in a Lagrangian formalism [37], it is sufficient to know the explicit form of $\underset{2}{g}{}_{00}$ and $\underset{3}{g}{}_{0k}$ (and not necessarily $\underset{4}{g}{}_{00}$) in order to write down the post-Newtonian equations of motion.

3. The arbitrariness in the choice of coordinates. Let us perform the coordinate transformation

$$x^0 = x'^0 + \underset{n+1}{a}{}^0\left(x'^\mu\right), \qquad x^k = x'^k + \underset{n}{a}{}^k\left(x'^\mu\right). \tag{24a}$$

The first terms affected by it are ($a_\alpha = \eta_{\alpha\beta}a^\beta$)

$$\begin{aligned} \underset{n+2}{g'}{}_{00} &= \underset{n+2}{g}{}_{00} + 2\underset{n+1}{a}{}_{0,0}, \\ \underset{n+1}{g'}{}_{0k} &= \underset{n+1}{g}{}_{0k} + \underset{n+1}{a}{}_{0,k} + \underset{n}{a}{}_{k,0}, \\ \underset{n}{g'}{}_{ik} &= \underset{n}{g}{}_{ik} + \underset{n}{a}{}_{i,k} + \underset{n}{a}{}_{k,i}. \end{aligned} \tag{24b}$$

It can easily be seen that if $(\underset{n}{g}{}_{ik}, \underset{n+1}{g}{}_{0k}, \underset{n+2}{g}{}_{00})$ is a solution of the field equations, then $(\underset{n}{g'}{}_{ik}, \underset{n+1}{g'}{}_{0k}, \underset{n+2}{g'}{}_{00})$ is also a solution of the same equations, representing the same physical situation in a different coordinate system. The *form* of the equations of motion considered as functions of the y's obviously depends on the coordinate system used. Similarly, in the ordinary geodesic equation

$$y''^\alpha + \Gamma^\alpha{}_{\mu\lambda}(y)y'^\mu y'^\lambda \equiv G^\alpha\left(y''^\mu, y'^\lambda, y^\beta\right) = 0,$$

the form of the function G^α depends on the coordinate system. More precisely the equations of motion of order $n+4$ ($n = 0, 2, \ldots$) depend on $\underset{n-1}{a}{}^0$ and $\underset{n}{a}{}^k$ (and also on coordinate changes of lower orders). The form of the Newtonian equations cannot be affected unless the Galilean character of $\underset{0}{g}{}_{\alpha\beta}$ is destroyed by the transformation. The post-Newtonian equations depend on the choice of $\underset{2}{a}{}^k$ in $\underset{2}{g}{}_{ik} = \delta_{ik}\underset{2}{g}{}_{00} + \underset{2}{a}{}_{i,k} + \underset{2}{a}{}_{k,i}$. The case $\underset{2}{a}{}_k = 0$ corresponds to the choice of harmonic coordinates in this approximation.

Sometimes doubts are raised as to the physical meaning of conclusions drawn from equations of motion which depend on the frame of reference. The answer to this objection is rather easy and can be made trivial by the following example: consider the curve $x_1 = \sin x_2$. This is a sine curve if the metric is $ds^2 = dx_1{}^2 + dx_2{}^2$ or a circle if $ds^2 = dx_1{}^2 + x_1{}^2 dx_2{}^2$. An equation of motion has no intrinsic meaning of its own. It is only the knowledge of equations of motion (or of a solution thereof) *and* of the corresponding metric which enables us to draw some physical (observational) conclusions, e.g., as to the advance of the periastron. In some special cases (e.g. static or periodic metric) it is not necessary to make explicit use of the form of the metric tensor.[5]

It is possible to simplify the equations of motion of a given order but only at the price of complicating the metric [28,32].

4. Radiation terms in the EIH method. The structure of Einstein's equations is such that we *can* choose solutions of the form

$$\begin{aligned} g_{00} &= 1 + \underset{2}{g}_{00} + \underset{4}{g}_{00} + \underset{6}{g}_{00} + \dots, \\ g_{0k} &= \underset{3}{g}_{0k} + \underset{5}{g}_{0k} + \underset{7}{g}_{0k} + \dots, \\ g_{ik} &= -\delta_{ik} + \underset{2}{g}_{ik} + \underset{4}{g}_{ik} + \underset{6}{g}_{ik} + \dots \end{aligned} \tag{25}$$

By analogy with the scalar wave equation and Maxwell's theory we can interpret solutions of the form (25) as representing standing wave fields (no secular losses of energy by radiation). It is only these solutions which were considered in the classical papers on the EIH method [19–25]. In order to get solutions corresponding to "retarded" or "advanced" fields we must supplement the series (25) with the missing terms: odd in g_{00} and g_{ik} and even in g_{0k} ("radiation terms"). The first of these radiation terms satisfy linear *homogeneous* equations and we may expect they are linear in the masses and hence their form can be derived from the linearized theory. The electromagnetic analogy suggests that the first radiation terms depend only on time and apparently can be removed by a coordinate transformation (24a); e.g., if $\underset{5}{g}_{00} = f(t)$ and $\underset{4}{a}_0 = -\frac{1}{2}\int f(t)dt$ then $\underset{5}{g'}_{00} = 0$ [28–30]. However, the whole field $(\underset{n}{g}_{ik}, \underset{n+1}{g}_{0k}, \underset{n+2}{g}_{00})$ can be annihilated by means of (24a) when and only when the following conditions are satisfied:

$$\underset{n+2}{g}_{00,ik} + \underset{n}{g}_{ik,00} - \underset{n+1}{g}_{i0,k0} - \underset{n+1}{g}_{k0,i0} = 0,$$

[5] Some of these remarks are due to discussions with Dr.W. Tulczyjew. The topics raised in this lecture are thoroughly discussed in a monograph on the problem of motion in general relativity which is being prepared by Professor L. Infeld and Dr. J. Plebański. [Note added by the Editor: The reference is: L. Infeld, J. Plebański, *Motion and relativity*. PWN and Pergamon Press, Warsaw and Oxford 1960.]

$$\underset{n+1}{g}{}_{0m,ik} + \underset{n}{g}{}_{ik,0m} - \underset{n+1}{g}{}_{0i,km} - \underset{n}{g}{}_{km,0i} = 0, \tag{26}$$

$$\underset{n}{g}{}_{im,kl} + \underset{n}{g}{}_{kl,im} - \underset{n}{g}{}_{il,km} - \underset{n}{g}{}_{km,il} = 0.$$

That is to say, equations (26) constitute a system of necessary and sufficient conditions for the existence of functions $\underset{n+1}{a}{}_0$ and $\underset{n}{a}{}_k$ such that $\underset{n}{g'}{}_{ik} = \underset{n+1}{g'}{}_{0k} = \underset{n+2}{g'}{}_{00} = 0$. It was remarked by Goldberg [31] that starting with $\underset{n}{g}{}_{ik} = f_{ik}(t)$ we can choose solutions of the field equations in the $(n+1)$th and $(n+2)$th orders such that the conditions (26) will not be satisfied. However it must be noted that since the solutions of the field equations are non unique, we can also start with the same $\underset{n}{g}{}_{ik}$ and obtain functions $\underset{n+1}{g}{}_{0k}$ and $\underset{n+2}{g}{}_{00}$ which can be annihilated. For example the field

$$\underset{n}{g}{}_{ik} = f_{ik}(t), \qquad \underset{n+1}{g}{}_{0k} = \tfrac{1}{2}x^s \dot{f}_{sk}, \qquad \underset{n+2}{g}{}_{00} = 0$$

is flat, but the field

$$\underset{n}{g}{}_{ik} = f_{ik}(t), \qquad \underset{n+1}{g}{}_{0k} = 0, \qquad \underset{n+2}{g}{}_{00} = -r^2 \ddot{f}_{ss}/6$$

is empty and non-flat unless $\ddot{f}_{ik} = \frac{1}{3}\delta_{ik}\ddot{f}_{ss}$ (spherical symmetry), namely

$$\underset{n+2}{g}{}_{00,ik} + \underset{n}{g}{}_{ik,00} - \underset{n+1}{g}{}_{i0,k0} - \underset{n+1}{g}{}_{k0,i0} = \ddot{f}_{ik} - \tfrac{1}{3}\delta_{ik}\ddot{f}_{ss}.$$

The exact form of the first radiation terms for a system of point particles can be obtained from (9). The *linear part* of $\underset{n}{g}{}_{\alpha\beta}$ is connected to $\underset{n}{\gamma}{}_{\alpha\beta}$ by the equation

$$\underset{n}{g}{}^{\text{linear}}_{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\lambda}\big(\underset{n}{\gamma}{}^{\mu\lambda} - \tfrac{1}{2}\eta^{\mu\lambda}\eta_{\pi\rho}\underset{n}{\gamma}{}^{\pi\rho}\big). \tag{27}$$

From (9) and (27) we have

$$\underset{3}{g}{}_{ik} = 0, \tag{28a}$$

$$\underset{4}{g}{}_{0k} = -\underset{4}{\gamma}{}^{0k} = -4\sum m\ddot{y}^k = 0. \tag{28b}$$

The last equality holds by virtue of the Newtonian equations of motion [26] and is to be read: $\sum m\ddot{y}^k$ is at least of order 6.

$$\underset{5}{g}{}_{00} = \frac{2}{3!}\sum \frac{d^3}{dt^3}mr^2 + 2\sum \frac{d}{dt}m\dot{y}^s\dot{y}^s. \tag{28c}$$

The field defined by (28) is trivial and can be annihilated by a coordinate transformation. It was shown by Infeld that this field does not contribute to the equations of motion of the 7th order. The next radiative set is $\left(\underset{5}{g}_{ik}, \underset{6}{g}_{0k}, \underset{7}{g}_{00}\right)$ and we have from the linearized theory:

$$\underset{5}{g}_{ik} = 4\sum \frac{d}{dt} m\dot{y}^i\dot{y}^k + \delta_{ik}\sum\left(\frac{1}{3}\frac{d^3}{dt^3}mr^2 - 2\frac{d}{dt}m\dot{y}^s\dot{y}^s\right), \tag{29a}$$

$$\underset{6}{g}_{0k} = -4\sum m\ddot{y}^k - \frac{4}{3!}\sum\frac{d^3}{dt^3}mr^2\dot{y}^k. \tag{29b}$$

The equation for $\underset{7}{g}_{00}$ is inhomogeneous and must be solved. We obtain

$$\begin{aligned}\underset{7}{g}_{00} = {}& \frac{2}{5!}\sum\frac{d^5}{dt^5}mr^4 + \frac{2}{3!}\sum\frac{d^3}{dt^3}mr^2\dot{y}^s\dot{y}^s + \underset{5}{g}_{ik}\sum mr_{,ik}\\ &+\left(\underset{5}{g}_{00} + \tfrac{1}{2}\underset{5}{g}_{ss}\right)\underset{2}{g}_{00}.\end{aligned} \tag{29c}$$

The fields (29) cannot be annihilated by a coordinate transformation.

5. Radiation damping in the problem of two bodies. Gravitational radiation does not occur in the 4th (Newtonian), 6th (post-Newtonian) and 8th (never explicitly evaluated) approximation orders, but it can occur in the 9th order. It is not possible to write down the equations of motion up to 9th order explicitly because we do not know the contributions of order 8 (the knowledge of $\underset{4}{g}_{ik}$, $\underset{5}{g}_{0k}$ and $\underset{6}{g}_{00}$ is needed for this). However, in some simple cases we can foresee the form of 8th order contributions and write down the 9th order corrections.

Let us take the example of two bodies of equal mass m that in the nonradiative approximations move uniformly along a circle of (coordinate) radius R. We choose the origin of the spatial coordinates at the centre of inertia and denote by $y^k(t)$ the coordinates of one of the bodies. The equations of motion up to 8th order have the form

$$\ddot{y}^k + \omega^2\left(m, y^s y^s, \dot{y}^k y^k, \dot{y}^m\dot{y}^m\right) y^k = 0$$

and admit solutions of the form

$$y^1 = R\cos\omega_0 t, \qquad y^2 = R\sin\omega_0 t, \qquad y^3 = 0. \tag{30}$$

The 9th order corrections can be evaluated from (23) using the field given by (29). It turns out that the equations of motion up to 9th order have in our case the form

$$\ddot{y}^k + 2\alpha\left(m, y^s y^s, \dot{y}^m y^m, \dot{y}^n\dot{y}^n\right)\dot{y}^k + \omega^2 y^k = 0. \tag{31}$$

Here α is of 6th order and is constant by virtue of (30); a laborious computation gives

$$\alpha = \frac{3}{20}\frac{m^3}{R^4}. \tag{32}$$

Table summarizing the information necessary to obtain the equations of motion from the "geodesic" equation (22)

Equations of motion of order	Orders of needed fields			these fields depend on		Equations of motion depend on		Orders of non-vanishing components of the Riemann tensor derivable from the corresponding field		
	g_{ik}	g_{ok}	g_{oo}	a_0	a_k	a_0	a_k	R_{iook}	R_{iokl}	R_{iklm}
4 (Newtonian)	0	1	2	1	0	–	0	2	–	–
6 (EIH)	2	3	4	3	2	1	2	4	3	2
8 (not evaluat.)	4	5	6	5	4	3	4	6	5	4
9 (1st radiat. corr.)	5	6	7	6	5	4	5	7	–	–
...	...	...	...	...	...	...	...	...	...	...

ω^2 contains a nonlinear term of second order so that the equation (31) is in reality nonlinear. The damping term $2\alpha\dot{y}^k$ excludes the possibility of periodic solutions. It is not easy to give a clear physical interpretation to this result; in particular we do not know if the diminishing of the *coordinate* distance between the particles (due to $\alpha > 0$) is accompanied by a decrease of the *geometrical* distance. However, it seems to be proved by this contribution that gravitational radiation induces secular changes in the motion of bodies.

LECTURE IV

THREE PROBLEMS OF GENERAL RELATIVITY

1. Propagation of gravitational disturbances. Einstein's field equations are (for physically acceptable metrics) of the hyperbolic type, and as such admit non-analytic solutions. The existence of such solutions is essential for the transmission of information [38]. Non-analytic functions can possess discontinuities in derivatives of a certain order and it is of some interest to study the form of these discontinuities. They can occur, for example, at the front of a wave and the knowledge of their behaviour can provide some information about the wave itself.

In electrodynamics we may assume that the discontinuities occur in the first derivatives of the electromagnetic field. It turns out that the discontinuities can take place only on null hypersurfaces. This means they must move with the velocity of light. Denoting by ΔF the jump of a field F across the hypersurface Σ defined by $f(x) = 0$, we have [14]

$$\Delta\mathbf{E}_{,\alpha} = \mathbf{e} f_{,\alpha}, \qquad \Delta\mathbf{H}_{,\alpha} = \mathbf{n} \times \mathbf{e} f_{,\alpha}, \qquad \mathbf{n} \cdot \mathbf{e} = 0,$$

where $\boldsymbol{n} = \text{grad } f/|\text{grad } f|$. The geometrical structure of the discontinuity resembles in this case that of the plane wave. It will be seen that the situation is similar in the theory of gravitation.

With Lichnerowicz [9] let us assume that the Riemannian space-time V_4 is such that there exist (at least locally) coordinate systems in which $g_{\mu\lambda}$ is of class C^1 and piecewise of class C^3. We shall restrict ourselves to these coordinate systems only; therefore the admissible coordinate transformations will be of class (C^2, C^4 piecewise). The discontinuities of $g_{\mu\lambda,\alpha\beta}$, across Σ ($f = 0$) can be written in the form [14, 39]:

$$\Delta g_{\mu\lambda,\alpha\beta} = h_{\mu\lambda} f_{,\alpha} f_{,\beta}. \tag{1}$$

By virtue of the assumptions on the differentiable structure of V_4 the functions $h_{\mu\lambda}$ and

$$h'_{\mu\lambda} = h_{\mu\lambda} + h_\mu f_{,\lambda} + h_\lambda f_{,\mu}, \qquad (h_\mu = \text{arbitrary}), \tag{2}$$

represent (geometrically) the same discontinuity [40]. If $\Delta R_{\mu\lambda\alpha\beta} = 0$, then one can choose h_μ so as to obtain $h'_{\mu\lambda} = 0$; in this case the discontinuities have no physical meaning and are due to the coordinate system. Assuming the empty space-time equations $R_{\mu\lambda} = 0$ we obtain some conditions on $h_{\mu\lambda}$, namely

$$g^{\alpha\beta}\left(h_{\mu\lambda} f_{,\alpha} f_{,\beta} + h_{\alpha\beta} f_{,\mu} f_{,\lambda} - h_{\mu\alpha} f_{,\lambda} f_{,\beta} - h_{\lambda\beta} f_{,\mu} f_{,\alpha}\right) = 0. \tag{3}$$

If $g^{\alpha\beta} f_\alpha f_{,\beta} \neq 0$ then (3) implies

$$h_{\mu\lambda} = a_\mu f_{,\lambda} + a_\lambda f_{,\mu}$$

and the discontinuity is spurious. "True" discontinuities can appear only on null hypersurfaces; in this case equation (3) is equivalent to [40]

$$(h_\mu{}^\nu - \tfrac{1}{2}\delta_\mu{}^\nu h_\tau{}^\tau) f_{,\nu} = 0, \qquad g^{\rho\sigma} f_{,\rho} f_{,\sigma} = 0 \tag{4}$$

and $\Delta R_{\mu\lambda\alpha\beta}$ can be put in the form [13]

$$\Delta R_{\mu\lambda\alpha\beta} = m_{\mu\lambda} m_{\alpha\beta} - n_{\mu\lambda} n_{\alpha\beta} \tag{5}$$

(m and n are simple null bivectors) corresponding to type II with vanishing scalar invariants [12, 41], (lecture I). The local geometry of a gravitational disturbance is thus the same as the local geometry of a plane wave.

Equations (3) or (4) constitute some algebraic conditions which must be fulfilled by the discontinuities. However, if a field of discontinuities $h_{\mu\lambda}$ is given on a 2-surface S lying on a space-like σ then its further propagation is determined by the field equations. It follows from this argument that the conditions (4) should be supplemented by some differential equations describing the evolution of $h_{\mu\lambda}$ in time. We shall now derive these equations and apply them to study the propagation of discontinuities in Schwarzschild space-time [42].

Let us assume that $f = \text{const}$ is the equation of a family of null hypersurfaces and Σ $(f = 0)$ is one of them. The curves $x^\mu = x^\mu(\lambda)$ defined on Σ by

$$dx^\alpha/d\lambda = g^{\alpha\beta} f_{,\beta} \tag{6}$$

are null geodesics (they are bicharacteristics of Einstein's equations) and it is possible to obtain an equation describing the behaviour of the discontinuities along these "gravitational rays." Let us take a coordinate system in which $f \equiv x^0$ (thus $g^{00} \equiv 0$) and calculate $\Delta R_{ik,0} = 0$. These are equations we are seeking for, but written in a non-covariant form. We can find the covariant equations imposing on them the following conditions:

1) they should reduce to $\Delta R_{ik,0} = 0$ when $f \equiv x^0$;

2) they should determine $h_{\mu\lambda}$ only up to a transformation (2).

The result is

$$2\frac{D}{d\lambda}\Delta R_{\mu\lambda\alpha\beta} + \Box f \Delta R_{\mu\lambda\alpha\beta} = 0. \tag{7}$$

where

$$\Box f = g^{\mu\lambda} f_{;\mu\lambda}, \qquad \Delta R_{\mu\lambda\alpha\beta} = \tfrac{1}{2} f_{,[\mu} h_{\lambda][\alpha} f_{,\beta]}.$$

The following properties of (7) are of interest: these ordinary differential equations are linear and homogeneous in $\Delta R_{\mu\lambda\alpha\beta}$; thus if $\Delta R_{\mu\lambda\alpha\beta} = 0$ at a point of the curve $x^\alpha = x^\alpha(\lambda)$, then $\Delta R_{\mu\lambda\alpha\beta}$ vanishes along the whole curve. If the algebraic conditions (3) are satisfied on an initial surface S, then they will be satisfied by virtue of (7) on the whole of Σ. If Σ is harmonic $(\Box f = 0)$ then the tensor $\Delta R_{\mu\lambda\alpha\beta}$ is parallelly propagated.

As an example, let us consider the propagation of discontinuities in a space-time which initially possessed the Schwarzschild metric

$$ds^2 = (1 - 2m/r)dt^2 - dr^2/(1 - 2m/r) - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{8}$$

At the time $t = 0$ on the surface $r = r_0$ there appears a discontinuity characterized by $\Delta R_{\mu\lambda\alpha\beta}(0, r_0, \theta, \phi) = \Delta R_{\mu\lambda\alpha\beta}(r_0)$; it will propagate along a hypersurface Σ with equation

$$f = t - F(r) = 0, \qquad g^{\mu\lambda} f_{,\mu} f_\lambda = 0, \qquad F(r_0) = 0.$$

We find $F(r)$ to be

$$F(r) = r - r_0 + 2m \log\frac{r - 2m}{r_0 - 2m}, \qquad (r > r_0 > 2m).$$

Solving (6) for the metric given by (8) one obtains the 2-parameter family of geodesics:

$$t = r - r_0 + 2m \log\frac{r - 2m}{r_0 - 2m}, \qquad \theta = \text{const}, \qquad \phi = \text{const},$$

(r is used instead of the parameter λ). The equations (7) can now easily be solved; it is convenient to express the result in terms of the physical components [43] of $\Delta R_{\mu\lambda\alpha\beta}$. We introduce the tetrads of orthonormal vectors $\lambda^{\mu}{}_{\underline{\alpha}}$ ($\underline{\alpha}, \underline{\beta}, \ldots = \underline{0}, \underline{1}, \underline{2}, \underline{3}$ label the vectors) as follows:

$$\lambda^{\mu}{}_{\underline{0}} = \left(1/\sqrt{1-2m/r}, 0, 0, 0\right),$$

$$\lambda^{\mu}{}_{\underline{1}} = \left(0, \sqrt{1-2m/r}, 0, 0\right),$$

$$\lambda^{\mu}{}_{\underline{2}} = (0, 0, 0, 1/r, 0),$$

$$\lambda^{\mu}{}_{\underline{3}} = (0, 0, 0, 1/r \sin\theta).$$

The physical components of $\Delta R_{\mu\lambda\alpha\beta}$ are defined by

$$\Delta R_{\underline{\mu}\underline{\lambda}\underline{\alpha}\underline{\beta}} = \Delta R_{\mu\lambda\alpha\beta}\lambda^{\mu}{}_{\underline{\mu}}\lambda^{\lambda}{}_{\underline{\lambda}}\lambda^{\alpha}{}_{\underline{\alpha}}\lambda^{\beta}{}_{\underline{\beta}}.$$

The result of the calculation is

$$\Delta R_{\underline{\mu}\underline{\lambda}\underline{\alpha}\underline{\beta}}(r) = \Delta R_{\mu\lambda\alpha\beta}(r_0)(r_0 - 2m)/(r - 2m). \tag{9}$$

It is worth noting that $\Delta R_{\underline{\mu}\underline{\lambda}\underline{\alpha}\underline{\beta}}$ behaves like r^{-1} for large values of r; this result seems to confirm to some extent the general hypothesis about gravitational radiation formulated in the first lecture.

2. Conservation laws and symmetry; properties of space-time. A Lorentz-covariant field theory in flat space-time possesses 10 conservation laws which correspond to the 10-parameters group of motions of Minkowski space-time. In general relativity one can formulate some conservation laws involving the pseudotensor of energy and momentum of the gravitational field. The physical meaning of these laws is that the energy of matter and the electromagnetic field can be transformed into the gravitational energy and vice-versa; the "physical" energy of matter alone is not conserved. However, if the space-time admits a group of motions, then it is possible to find some covariant conservation laws, not involving the pseudotensor of the gravitational field. If v^{α} is a generator of a group of motions, i.e.

$$v_{\mu;\lambda} + v_{\lambda;\mu} = 0 \tag{10}$$

and $T^{\alpha\beta}$ is the energy-momentum tensor of matter, then [22]

$$\left(\underline{T}^{\alpha\beta} v_{\beta}\right)_{,\alpha} = \underline{T}^{\alpha\beta}{}_{;\alpha} v_{\beta} + \underline{T}^{\alpha\beta} v_{\alpha;\beta} = 0. \tag{11}$$

The number of these *conservation laws of matter* is equal to the number of parameters of the group of motions [10]. There are 10 laws of the form (11) only in spaces of constant curvature [22]. If the matter field is conform-invariant (which means $T = T_\mu{}^\mu = 0$) then the equation (11) gives a conservation law also in the more general case when v_α represents the generator of a group of conformal transformations, i.e., when v_α satisfies [44]

$$v_{\mu;\lambda} + v_{\lambda;\mu} = 2\alpha g_{\mu\lambda}. \tag{12}$$

As an example, we can take a flat space-time and the Maxwell field [45]. Equations (12) in Minkowski space-time have 15 independent solutions: 10 motions and 5 infinitesimal conformal transformations which are not motions (as generators one can take: $v^\alpha = ex^\alpha$, $v^\alpha = 2e_\beta x^\beta x^\alpha - e^\alpha x_\beta x^\beta$).

It is easy to write down the conservation laws in a form corresponding to the canonical laws of special relativity [46]. Let ψ denote a physical field (not $g_{\mu\lambda}$) and $\underline{L}(\psi, \psi_{,\alpha}, g_{\mu\lambda})$ the corresponding Lagrangian density, supposed to be a form-invariant function of its arguments. If $\delta^*\psi$ is the "substantial" variation of ψ corresponding to the infinitesimal transformation $x'^\alpha = x^\alpha + v^\alpha$, i.e. $\delta^*\psi = \psi'(x) - \psi(x)$, then the *vector density*

$$\underline{I}^\mu = \underline{L} v^\mu + \delta^*\psi \partial \underline{L}/\partial \psi_{,\mu} \tag{13}$$

is divergence-free if v^α satisfies (12) and $\alpha T = 0$ [47]. All these conservation laws are "weak," i.e. they hold when the free field equations for ψ are satisfied.

It is well known also that the number of independent first integrals of the equations of geodesics is equal to the number of parameters of the group of motions [44]. If $x^\mu = x^\mu(s)$ is a geodesic then

$$v^\alpha dx^\alpha / ds = \text{const}.$$

A generalization of this theorem to the case of particles interacting with physical (electromagnetic, scalar) fields is given in [10]. For null geodesics the expression $v^\alpha dx^\alpha / d\lambda$ is a first integral also in the case when v_α generates a conformal transformation.

3. The "fast" approximation method. As has been said before, the EIH method is not well suited to the investigation of gravitational radiation. It is therefore necessary to have recourse to another method of approximation, in which the time is treated on the same footing as the space coordinates. This "fast" or "old" approximation method was used by Einstein as early as in 1916 [48]. Einstein assumed that the field is weak and can be written in the form $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and that terms nonlinear in $h_{\mu\lambda}$ may be neglected in the field equations. This approach constitutes essentially the first step of an approximation method, which can be continued further. Namely, we can assume the expansion

$$g_{\alpha\beta} = \eta_{\alpha\beta} + k \underset{1}{h}_{\alpha\beta} + k^2 \underset{2}{h}_{\alpha\beta} + \dots,$$

where k is a parameter which may be identified with the gravitational constant or with some characteristic mass in the problem [49]. If we write Einstein's equations in the form

$$G_\mu{}^\lambda = -\kappa T_\mu{}^\lambda, \qquad \kappa = 8\pi k \tag{14}$$

and assume the expansions of $T_\mu{}^\lambda$ and $G_\mu{}^\lambda$

$$T_\mu{}^\lambda = \underset{0}{T}{}_\mu{}^\lambda + k\underset{1}{T}{}_\mu{}^\lambda + \dots, \quad G_\mu{}^\lambda = k\underset{1}{G}{}_\mu{}^\lambda + k^2\underset{2}{G}{}_\mu{}^\lambda + \dots \quad \left(\underset{0}{G}{}_\mu{}^\lambda \equiv 0\right),$$

then (14) becomes

$$\underset{1}{G}{}_\mu{}^\lambda = -8\pi\underset{0}{T}{}_\mu{}^\lambda, \quad \underset{2}{G}{}_\mu{}^\lambda = -8\pi\underset{1}{T}{}_\mu{}^\lambda, \qquad \dots \tag{15}$$

Einstein restricted himself to the first of equations (15). Fock [11], [22] and Bonnor [49] found some partial and special solutions of the second order equations. We shall now briefly discuss the theory of the first order approximation in k, which is known as

A) The linearized theory of gravitation. In many textbooks on general relativity this theory is presented in connection with the problem of gravitational waves and radiation. It seems important to realize to what extent this theory is different from, and which of its results have their counterparts in, Einstein's theory of gravitation. In this section we shall drop the index below $\underset{1}{h}{}_{\mu\lambda}$, and write $H_\mu{}^\lambda$ instead of $\underset{1}{G}{}_\mu{}^\lambda$ and $U_\mu{}^\lambda$ instead of $\underset{0}{T}{}_\mu{}^\lambda$. The field equations of the linearized theory become

$$H_\mu{}^\lambda \equiv S_\mu{}^\lambda - \tfrac{1}{2}\delta_\mu{}^\lambda S = -8\pi U_\mu{}^\lambda \tag{16}$$

where $S = S_\lambda{}^\lambda$ and

$$S_{\mu\lambda} = \eta^{\alpha\beta} S_{\mu\alpha\beta\lambda},$$

$$S_{\mu\lambda\alpha\beta} = \tfrac{1}{2}\left(h_{\mu\beta,\lambda\alpha} + h_{\lambda\alpha,\mu\beta} - h_{\mu\alpha,\lambda\beta} - h_{\lambda\beta,\mu\alpha}\right). \tag{16a}$$

The indices are raised and lowered by means of the Minkowski eta. The field equations are invariant with respect to the gauge transformations

$$h_{\mu\lambda} \to h'_{\mu\lambda} = h_{\mu\lambda} + a_{\mu,\lambda} + a_{\lambda,\mu} \tag{17}$$

and can be derived from a variational principle. For the Lagrangian density of the free field we can take

$$H = \tfrac{1}{2}\left(h_{\mu\lambda,\alpha}h^{\alpha\lambda,\mu} - h_{\mu\lambda}{}^{,\lambda}h_\alpha{}^{\alpha,\mu} + \tfrac{1}{2}h_\mu{}^\mu{}_{,\lambda}h_\alpha{}^{\alpha,\lambda} - \tfrac{1}{2}h_{\mu\lambda,\alpha}h^{\mu\lambda,\alpha}\right) \tag{18}$$

H is not invariant with respect to (17), but transforms according to the law $H \to H' = H + Q^{\beta}{}_{,\beta}$ [50]. From this follow the "Bianchi" identities

$$H^{\mu\lambda}{}_{,\lambda} \equiv 0 \tag{19}$$

and the existence of superpotentials $V_{\mu}{}^{\alpha\beta} = -V_{\mu}{}^{\beta\alpha}$ such that

$$H^{\alpha\beta} = V^{\alpha\lambda\beta}{}_{,\lambda}, \qquad V_{\mu}{}^{\nu\lambda} = \tfrac{1}{2}\eta^{\alpha[\beta}\delta_{\mu}{}^{\nu}\eta^{\lambda]\tau}h_{\alpha\beta,\tau}. \tag{20}$$

It is not possible to form a gauge-invariant function of $h_{\mu\lambda}$ and $h_{\mu\lambda,\alpha}$ alone. The equation $S_{\mu\lambda\alpha\beta} = 0$ is a necessary and sufficient condition for the existence of functions a_{μ} such that $h'_{\mu\lambda} = 0$. The 20 functions $S_{\mu\lambda\alpha\beta}$ have essentially the same properties as the 20 components of the Riemann tensor:

$$S_{\mu\lambda\alpha\beta} = S_{\alpha\beta\mu\lambda} = -S_{\lambda\mu\alpha\beta}, \qquad S_{\mu[\lambda\alpha\beta]} = 0, \qquad S_{\mu\lambda[\alpha\beta,\tau]} = 0. \tag{21}$$

The necessary conditions (21) are also locally sufficient for the existence of $h_{\alpha\beta}$ such that (16a) is true. This theorem seems to be connected with the problem of finding the metric for a given curvature tensor. The equations (16) and (21) are analogous to Maxwell's equations without potentials: $f^{\mu\lambda}{}_{,\lambda} = -4\pi j^{\mu}$, $f_{[\mu\lambda,\alpha]} = 0$. In the linearized theory it is possible to solve the equations for the "field" $S_{\mu\lambda\alpha\beta}$, without any reference to the "potentials" $h_{\mu\lambda}$.

However, it is easier to normalize the potentials by means of the Einstein–de-Donder condition

$$\gamma^{\mu\lambda}{}_{,\lambda} = 0, \tag{22a}$$

where

$$\gamma_{\mu\lambda} = h_{\mu\lambda} - \tfrac{1}{2}\eta_{\mu\lambda}\eta^{\alpha\beta}h_{\alpha\beta}, \tag{22b}$$

and then write the field equations in the form

$$\Box\gamma^{\mu\lambda} = 16\pi U^{\mu\lambda}. \tag{23}$$

$U^{\mu\lambda}$ has to satisfy the conservation law

$$U^{\mu\lambda}{}_{,\lambda} = 0. \tag{24}$$

Under some reasonable assumptions about $U^{\mu\lambda}$, the retarded solution of (23) satisfies equations (22a). The 4 continuity equations impose some conditions on the source of the field; for example, pole particles interacting with the $h_{\mu\lambda}$ field have to move uniformly along straight lines. The "equations of motion" for singularities can be obtained from the surface integrals

$$\frac{d}{dt}\oint V_{\mu}{}^{0k} n_k dS = 0.$$

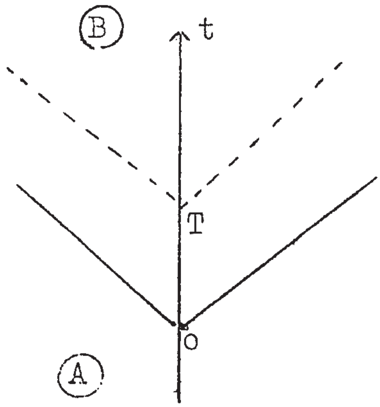

Figure 2.

However, the conditions (24) do not exclude the possibility of "wave solutions", depending on arbitrary functions of time. For example, if we take a quadrupole source [51, 52]

$$U^{00} = \alpha^{kl}\delta_{,kl}, \qquad U^{0k} = -\dot{\alpha}^{kl}\delta_{,l}, \qquad U^{kl} = \ddot{\alpha}^{kl}\delta, \tag{25}$$

where $\alpha^{kl} = \alpha^{lk}(t)$, then (24) is satisfied for arbitrary $\alpha^{kl}(t)$. Einstein and Eddington calculated the retarded field corresponding to (25) and, introducing it into the energy-momentum pseudotensor, evaluated the total energy radiated by these "gravitational waves." However, it is necessary to be very cautious in interpreting the results obtained by this method. Indeed, α^{kl} can be a periodic function and by the Einstein-Eddington method we obtain in this case a permanent outflow of radiation. On the other hand, it is obvious that a periodic metric excludes the possibility of secular changes which accompany a permanent outgoing wave. Periodic gravitational fields can describe standing-wave processes only. Further, we can regard $\eta_{\mu\lambda} + kh_{\mu\lambda}$ (where $h_{\mu\lambda}$ is calculated from (22b), (23) and (25)) as an exact metric of a space-time filled with matter described by $\theta_{\mu\lambda} \equiv -\kappa^{-1}G_{\mu\lambda}\left[\eta_{\alpha\beta} + kh_{\alpha\beta}\right]$. In this case the total radiated energy and momentum will be defined by the time integral of the flux of $\underline{\theta}_{\mu}{}^{k} + \underline{t}_{\mu}{}^{k}$ through a large sphere and can be shown to be equal to zero. In order to draw some physical conclusions it seems necessary to pass to

B) higher approximations. If the $\underset{1}{h}_{\mu\lambda}$ field really represents an outgoing gravitational wave, one expects to find in the 2nd order a decrease of the total energy (mass) of a radiating system. For example, we may take a Schwarzschild field of mass m, "superimpose" on it the field due to (25) where α^{kl} is a pulse, i.e. a regular function vanishing outside the interval $0 < t < T$ and compare the initial mass m (region A) with the total final energy (region B, $t \to \infty$). It is not obvious that the metric must be a Schwarzschild one in the region B, but it seems plausible to assume that the metric in B is static, at least asymptotically for $t \to \infty$. If this is the case, it is possible to determine the mass in region B by investigating the $1/r$ terms in the metric. Supplementing $U^{\mu\lambda} = \underset{0}{T}{}_{\mu\lambda}$ given by the formulae (25) by a term representing a point mass m, it is possible to write the first order field in the form

$$\underset{1}{\gamma}{}_{00} = -4m/r - 4\left(\alpha^{kl}(t-r)/r\right)_{,kl},$$

$$\underset{1}{\gamma}{}_{0k} = -4\left(\dot{\alpha}^{kl}(t-r)/r\right)_{,l}, \qquad \underset{1}{\gamma}{}_{kl} = -\ddot{\alpha}^{kl}(t-r)/r. \tag{26}$$

$\underset{n}{\gamma}{}_{\mu\lambda}$ is related to $\underset{n}{h}{}_{\mu\lambda}$ by a formula of the same form as (22b). In the second order we can assume $\underset{1}{T}{}_{\mu\lambda} = 0$. Imposing on $\underset{2}{\gamma}{}_{\mu\lambda}$ the condition $\underset{2}{\gamma}{}^{\mu\lambda}{}_{,\lambda} = 0$ we can write the field equations in the symbolical form

$$\Box \underset{2}{\gamma} = \underset{1}{\gamma} \cdot \underset{1}{\gamma}\,. \tag{27}$$

The right-hand side of this equation is a function quadratic in $\underset{1}{\gamma}{}_{\alpha\beta,\nu}$. Fock [11], [22] found an approximate solution of (26):

$$\underset{2}{\gamma}{}_{\mu\lambda} = F(\mathbf{n}, t-r)r^{-1}\log r k_\mu k_\lambda + \ldots, \tag{28}$$

where the dots stand for terms which for large r are small when compared with $\log r/r$, and F denotes a function with the following properties

$$F(\mathbf{n}, t) = \begin{cases} 0 & \text{for } \ t \leqslant 0, \\ F(\mathbf{n}) \geqslant 0 & \text{for } \ t > T. \end{cases}$$

$F(\mathbf{n})$ is proportional to the energy radiated by the system (as calculated from the pseudotensor) in a unit solid angle characterized by $\mathbf{n}$. However, it is not possible to evaluate the mass of the field given by (26), (28) in region B. The integral $\oint \underline{U}_0{}^{0k} n_k dS$ calculated up to the second order is divergent because of the $\log r/r$ term.

Bonnor [49] has attacked a similar problem by a method slightly different from the approach presented above. He assumes the axial symmetry of the radiating system (two particles connected by a spring) and introduces a non-harmonic coordinate system in which the metric is diagonal. The field $\underset{2}{h}{}_{\mu\lambda}$ found by Bonnor is time dependent in region B, but for large t contains only $1/r$ (and smaller) terms. The $\log r/r$ terms appear also in his calculation but only with nonsecular coefficients (vanishing in region B). The decrease of the gravitational mass defined by the $1/r$ terms is exactly equal to the total radiated energy, calculated from the pseudotensor.

C) Spherically symmetric scalar radiation field in general relativity. The method of approximation with respect to k can also be applied when the gravitational field interacts with some other physical fields. This can be illustrated by the example of a "model" scalar field ϕ satisfying the covariant wave equation

$$\left(\sqrt{-g}g^{\mu\lambda}\phi_{,\lambda}\right)_{,\mu} = 0. \tag{29}$$

Let us take the simplest case, namely that of a spherically symmetric field and assume

$$ds^2 = e^{\mu}dt^2 - e^{\lambda}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad \text{and}$$

$$\phi = \phi(r,t), \qquad \mu = \mu(r,t), \qquad \lambda = \lambda(r,t). \tag{30}$$

We shall find an approximate solution of (29) and of Einstein's equations, corresponding to an outgoing scalar wave. It will appear that the *gravitational mass* is diminished by an amount equal to the total energy carried out by the ϕ-field. The energy-momentum tensor for the ϕ-field was given in lecture I. Einstein's equations and the wave equation (29) become

$$e^{-\lambda}\left(\lambda'/r - 1/r^2\right) + 1/r^2 = k\left(e^{-\lambda}\phi'^2 + e^{-\mu}\dot{\phi}^2\right), \tag{31a}$$

$$e^{-\lambda}\left(\mu'/r + 1/r^2\right) - 1/r^2 = k\left(e^{-\lambda}\phi'^2 + e^{-\mu}\dot{\phi}^2\right), \tag{31b}$$

$$e^{-\lambda}\dot{\lambda}/r = 2ke^{-\lambda}\dot{\phi}\phi', \tag{31c}$$

$$\left(r^2e^{(\lambda-\mu)/2}\dot{\phi}\right)^{\cdot} - \left(r^2e^{-(\lambda-\mu)/2}\phi'\right)' = 0, \tag{31d}$$

where the dot and the prime denote, respectively, the derivatives with respect to t and r. The total energy contained in the field given by (30), where $\lambda_{,\mu} = O(r^{-1})$, is equal to

$$P_0 = \oint U_0{}^{0k}n_k dS = \lim_{r\to\infty} r\lambda/(2k).$$

The radiated power is

$$W_0 = \dot{P}_0 = \lim_{r\to\infty} r\dot{\lambda}/(2k) = \lim_{r\to\infty} r^2\dot{\phi}\phi'.$$

The last equality holds by virtue of (31c). Assuming the expansions

$$\phi = \underset{0}{\phi} + k\underset{1}{\phi} + \ldots, \qquad \lambda = k\underset{1}{\lambda} + k^2\underset{2}{\lambda} + \ldots,$$

$$\mu = k\underset{1}{\mu} + k^2\underset{2}{\mu} + \ldots$$

one gets the linearized equations

$$r \underset{1}{\lambda}' + \underset{1}{\lambda} = r^2 \big(\dot{\underset{0}{\phi}}^2 + \underset{0}{\phi}'^2\big), \tag{32a}$$

$$r \underset{1}{\mu}' - \underset{1}{\lambda} = r^2 \big(\dot{\underset{0}{\phi}}^2 + \underset{0}{\phi}'^2\big), \tag{32b}$$

$$\dot{\underset{1}{\lambda}} = 2r\, \dot{\underset{0}{\phi}}\, \underset{0}{\phi}', \tag{32c}$$

$$(r^2 \dot{\underset{0}{\phi}})^{\cdot} - (r^2 \underset{0}{\phi}')' = 0. \tag{32d}$$

A possible solution of (32d) is $\phi_0 = a(t-r)/r$, where $a(t)$ is a regular "pulse" function (vanishing for $t < 0$ and $t > T$). The general solution of (32a) and (32c) is

$$\underset{1}{\lambda} = 2m/r - (2/r) \int_0^{t-r} \dot{a}^2(t')dt' - a^2(t-r)/r.$$

The system of coordinates defined by (30) is determined to within a transformation of the time: $t \to t' = f(t)$. Accordingly, the solution for μ will contain an arbitrary function of time. We can choose it in such a way as to obtain a time-independent metric in region A:

$$\underset{1}{\mu} = -2m/r + (2/r) \int_0^{t-r} \dot{a}^2(t')dt'$$

$$-2 \int_0^{t-r} \Big(2\dot{a}^2(t')(t-t')^{-1} + a(t')\dot{a}(t')(t-t')^{-2}\Big)\, dt'.$$

Finally, we have in region A ($t - r < 0$):

$$\underset{1}{\lambda} = 2m/r, \qquad \underset{1}{\mu} = -2m/r,$$

and in region B ($t - r > T$):

$$\underset{1}{\lambda} = 2(m - \Delta m)/r, \qquad \underset{1}{\mu} = -2(m - \Delta m)/r + \text{function of time alone},$$

where

$$\Delta m = \int_0^T \dot{a}^2(t')dt'.$$

The first order field in region B corresponds also to a Schwarzschild field, because the function of time in $\underset{1}{\mu}$ can be absorbed by a transformation $t \to t'$. However, it is not possible to find a single coordinate system, in which the metric has the form (30) and is time-independent in both region A and B.

LECTURE V

EQUATIONS OF MOTION OF ROTATING BODIES[6]

The first papers on the equations of motion dealt only with the problem of spherically-symmetric, non-rotating bodies, described by "pole-particles" in the method of singularities. If one wants to obtain the motion and the field due to bodies with given internal structure, one must introduce higher poles, forbidden by the original EIH prescriptions. The first and the simplest question which arises concerns the motion of test particles with internal degrees of freedom (angular momentum, quadrupole momentum etc.). This problem has been discussed by a special approximate method by Mathisson [54] and Lubański [55], and in general relativity by Papapetrou [56]. The approach presented here has the advantage of being relativistically invariant (the derivation of Papapetrou is not) and applicable to particles with arbitrarily high multipole structure. The motion of heavy rotating bodies is discussed in Fock's book [22] and in a paper by Haywood [57], who, however, neglected some terms of order l/L in the equations of motion [7]. Haywood's equations differ only by non-essential terms from the EIH equations for pole-particles. The post-Newtonian equations containing the corrections of order l/L due to rotation have been found by Tulczyjew [53]. Starting from these equations it is possible to derive a new relativistic effect consisting of the precession of the plane of revolution.

1. Representation of extended bodies by means of singularities. Let us first take a scalar (Newtonian) potential ϕ, satisfying the Poisson equation

$$\Delta\phi = -4\pi f, \tag{1}$$

where f denotes a regular function, vanishing outside a bounded region whose dimensions are of order l. The solution of (1) can be written as

$$\phi(\mathbf{r}) = \int f(\mathbf{r}')R^{-1}dV'. \tag{2}$$

[6] This lecture is based mainly on the work of W. Tulczyjew [36, 53]. I take responsibility for this presentation of the results.

[7] As before, l is a length characterizing the dimensions of the bodies and L the distance between them.

Expanding $R^{-1} = |\mathbf{r} - \mathbf{r}'|^{-1}$ into a power series around the point $\mathbf{r}' = 0$ it is possible to write ϕ in the form

$$\phi(\mathbf{r}) = \mu r^{-1} - \mu^i (1/r)_{,i} + (1/2!)\mu^{ik}(1/r)_{,ik} - \ldots, \tag{3}$$

where the coefficients $\mu, \mu^i, \ldots$ are given by

$$\mu = \int f dV, \qquad \mu^i = \int x^i f dV, \qquad \mu^{ik} = \int x^i x^k f dV, \qquad \ldots,$$

$\mu^{i_1 \ldots i_n}/\mu$ being of the order l^n. Neglecting the quadrupole field is equivalent to treating l^2/r^2 as small. If $\mu \neq 0$ all the higher moments depend on the choice of the origin of coordinates which may always be localized in the centre of mass. The dipole or static moment μ^i vanishes in this case. If $\mu = 0$ then μ^i does not depend on the choice of the origin but this case does not occur in the theory of gravitation. The series (3) represents the field only outside the body and in general is divergent for small values of r.

In the δ-functions formalism, the ϕ given by (3) is a solution of

$$\Delta\phi = -4\pi \left(\mu\delta - \mu^i \delta_{,i} + \tfrac{1}{2}\mu^{ik}\delta_{,ik} - \ldots\right)$$

and we can write *symbolically*

$$f = \mu\delta - \mu^i \delta_{,i} + \tfrac{1}{2}\mu^{ik}\delta_{,ik} - \ldots \tag{4}$$

($\delta = \delta(\mathbf{r})$ denotes the three-dimensional Dirac function). Equation (4) means only that the *exterior* field due to f is equal to a sum of harmonic fields associated with $\mu\delta$, $-\mu^i\delta_{,i}$, etc. Equation (4) becomes meaningful when one integrates its both sides with $x^i x^k \ldots x^n$ (equality of momenta).

Every exterior static Newtonian field can be thus described by a denumerable set of coefficients (the "gravitational skeleton" of Mathisson).

It is not quite obvious that a gravitational skeleton exists for a given body in general relativity. We shall assume that it does, or at least we shall confine the discussion to bodies for which can be found an "equivalent" energy-momentum tensor built from δ-functions. This energy-momentum tensor will be assumed to have the form

$$\begin{aligned} \underline{T}^{\alpha\beta} = \sum \int_{-\infty}^{\infty} ds \big[\mu^{\alpha\beta}\delta_{(4)} - \left(\mu^{\alpha\beta\lambda_1}\delta_{(4)}\right)_{;\lambda_1} + \ldots \\ + (-1)^k (k!)^{-1} \left(\mu^{\alpha\beta\lambda_1 \ldots \lambda_k}\delta_{(4)}\right)_{;\lambda_1 \ldots \lambda_k} \big], \end{aligned} \tag{5}$$

where the sum is extended over all bodies, $\delta_{(4)} = \delta_{(4)}\left(x^\lambda - y^\lambda(s)\right)$ is the 4- dimensional Dirac's function, and the μ's are some tensor fields defined along the world lines and depending on s.

2. Equations of motion for test particles. The equations of motion of particles with a given structure, for example pole-dipole particles, can be obtained from $\underline{T}^{\alpha\beta}{}_{;\beta} = 0$ with $\underline{T}^{\alpha\beta}$ given by (5). The following two lemmas proved by Tulczyjew simplify the derivation of the equations of motions:

Lemma 1. *For every field $a^{\alpha\ldots\lambda}(s)$ regular along the line $y^\lambda = y^\lambda(s)$ we have the identity*

$$\int_{-\infty}^{\infty} ds \left(a^{\alpha\ldots\lambda} y'^{\mu} \delta_{(4)}\right)_{;\mu} \equiv \int_{-\infty}^{\infty} ds \delta_{(4)} D a^{\alpha\ldots\lambda}/ds, \qquad y'^{\lambda} = dy^\lambda/ds.$$

Lemma 2. *Every expression*

$$\underline{N}^{\alpha\ldots\beta} = \int_{-\infty}^{\infty} ds\left[\nu^{\alpha\ldots\beta}\delta_{(4)} + \left(\nu^{\alpha\ldots\beta|\lambda_1}\delta_{(4)}\right)_{;\lambda_1} + \ldots + \left(\nu^{\alpha\ldots\beta|\lambda_1\ldots\lambda_k}\delta_{(4)}\right)_{;\lambda_1\ldots\lambda_k}\right] \tag{6}$$

can be transformed into the "normal" form

$$\underline{N}^{\alpha\ldots\beta} = \int_{-\infty}^{\infty} ds\left[n^{\alpha\ldots\beta}\delta_{(4)} + \left(n^{\alpha\ldots\beta|\lambda_1}\delta_{(4)}\right)_{;\lambda_1} + \ldots + \left(n^{\alpha\ldots\beta|\lambda_1\ldots\lambda_k}\delta_{(4)}\right)_{;\lambda_1\ldots\lambda_k}\right],$$

where the n's are symmetric in the λ's and orthogonal to y'^{μ}:

$$n^{\alpha\ldots\beta|\lambda_1\ldots\lambda_p} = n^{\alpha\ldots\beta|(\lambda_1\ldots\lambda_p)} \qquad and$$
$$n^{\alpha\ldots\beta|\lambda_1\ldots\lambda_p} y'_{\lambda_1} = 0.$$

The vanishing of all the n's is a necessary and sufficient condition for the vanishing of $\underline{N}^{\alpha\ldots\beta}$.

The proof of the first lemma is easy. The proof of the second lemma is based on the first lemma and on the formula expressing the skew part of the second covariant derivatives of a tensor. In order to prove that $\underline{N}^{\alpha\cdots\beta} = 0$ implies the vanishing of the n's we integrate the scalar density $K_{\alpha\ldots\beta}\underline{N}^{\alpha\ldots\beta} = 0$ ($K_{\alpha\ldots\beta}$ = arbitrary) over a 4-region and apply some kind of generalized Du Bois Raymond's lemma. The general procedure of obtaining the equations of motion is very simple; we take a $\underline{T}^{\alpha\beta}$ with a definite number of multipole terms (i.e., we fix k in the formula (5)) and write down $\underline{T}^{\alpha\beta}{}_{;\beta}$. This expression is of the type given by (6); one transforms it into the normal form and then requires the separate vanishing of all the coefficients n. As an example one can take a pole-dipole particle described by

$$\underline{T}^{\alpha\beta} = \int_{-\infty}^{\infty} ds\left[\mu^{\alpha\beta}\delta_{(4)} - \left(\mu^{\alpha\beta\lambda}\delta_{(4)}\right)_{;\lambda}\right]. \tag{7}$$

Without loss of generality it is possible to assume that *all* the μ's are orthogonal to the velocity in the λ-indices. In this case it means that

$$\mu^{\alpha\beta\lambda} y'_\lambda = 0.$$

$\mu^{\alpha\beta\lambda}$ can be written in the form

$$\mu^{\alpha\beta\lambda} = S^{\alpha\beta\lambda} + \tfrac{1}{2} S^{\alpha\lambda} y'^{\beta} + \tfrac{1}{2} S^{\beta\lambda} y'^{\alpha} + S^{\lambda} y'^{\alpha} y'^{\beta},$$

where the S are orthogonal to y'^{λ} and $S^{\alpha\beta\lambda} = S^{\beta\alpha\lambda}$. Similarly

$$\mu^{\alpha\beta} = m^{\alpha\beta} + m^{\alpha} y'^{\beta} + m^{\beta} y'^{\alpha} + m y'^{\alpha} y'^{\beta}$$

where

$$m^{\alpha\beta} = m^{\beta\alpha}, \qquad m^{\alpha\beta} y'_\beta = 0, \qquad m^{\alpha} y'_\alpha = 0.$$

S^{α} corresponds to the static (dipole) moment of the body and can be put equal to zero by an appropriate choice of the world line y. We shall assume in further work that $S^{\lambda} = 0$. By writing $T^{\alpha\beta}{}_{;\beta} = \int ds\big[\big(\mu^{\alpha\beta}\delta_{(4)}\big)_{;\beta} - \big(\mu^{\alpha\beta\lambda}\delta_{(4)}\big)_{;\lambda\beta}\big] = 0$ and applying the procedure outlined above one obtains the following set of equations

$$S^{\alpha\beta\lambda} + S^{\alpha\lambda\beta} + \tfrac{1}{2}(S^{\beta\lambda} + S^{\lambda\beta}) y'^{\alpha} = 0 \tag{8}$$

$$\text{or} \qquad S^{\beta\lambda} = -S^{\lambda\beta} \qquad \text{and} \qquad S^{\alpha\beta\lambda} = 0,$$

$$2m^{\alpha} = y'_\beta DS^{\beta\alpha}/ds, \tag{9a}$$

$$m^{\alpha\beta} = 0, \tag{9b}$$

$$DS^{\alpha\beta}/ds - y'^{\beta} y'_\lambda DS^{\alpha\lambda}/ds + y'^{\alpha} y'_\lambda DS^{\beta\lambda}/ds = 0, \tag{9c}$$

$$\frac{D}{ds}\left(m y'^{\alpha} + \frac{DS^{\lambda\alpha}}{ds} y'_\lambda\right) + \tfrac{1}{2} S^{\mu\lambda} y'^{\beta} R^{\alpha}{}_{\beta\mu\lambda} = 0. \tag{10}$$

These equations are essentially those obtained by Papapetrou [56]. The condition $S^{\alpha\beta} y'_\beta = 0$ imposed on $S^{\alpha\beta}$ by Pirani [43] is here satisfied ex definitione. It follows from equations (8)–(9b) that a pole-dipole particle is completely characterized by 4 parameters: the mass m (which is conserved by virtue of (9c), see [43]) and a space-like angular momentum vector [43]

$$H^{\mu} = \tfrac{1}{2} \eta^{\mu\lambda\alpha\beta} y'_\lambda S_{\alpha\beta}, \qquad H^{\mu} y'_\mu = 0,$$

where $\eta^{\mu\lambda\alpha\beta}$ is the alternating tensor. Equation (9c) governs the changes of angular momentum and can be rewritten in the form

$$DS^{\alpha\beta}/ds = \big(S^{\beta\lambda} y'^{\alpha} - S^{\alpha\lambda} y'^{\beta}\big) y''_\lambda.$$

Equation (10) describes the translatory motion of the particle and reduces to the geodesic equation for $S^{\alpha\beta} = 0$. From (9a) and (9b) one obtains the result enunciated without proof in one of the preceding lectures: for a simple pole particle $\mu^{\alpha\beta} = m y'^{\alpha} y'^{\beta}$.

The equations (8)–(10) can also be applied to heavy rotating bodies if one uses the "good-deltas" technique.

3. Post-Newtonian equations of motion of two heavy, rotating bodies. Let us consider the case of two bodies of finite mass $\overset{1}{m}$, $\overset{2}{m}$, with world lines described by $y^k = y^k(t)$ and $z^k = z^k(t)$. These bodies are supposed to possess some internal angular momentum and to have a pole-dipole structure (quadrupole effects being neglected). The equations of motion of these bodies can be obtained by an approximate method directly from (9c) and (10). The approximate equations can also be derived from the beginning without reference to (8)–(10).

The energy-momentum tensor density of the system can be written in the form

$$\underline{T}^{\alpha\beta} = \overset{1}{t}{}^{\alpha\beta}\overset{1}{\delta} - \overset{1}{t}{}^{\alpha\beta k}\overset{1}{\delta}{}_{,k} + \overset{2}{t}{}^{\alpha\beta}\overset{2}{\delta} - \overset{2}{t}{}^{\alpha\beta k}\overset{2}{\delta}{}_{,k} \tag{11}$$

$$\overset{1}{\delta} = \delta_{(3)}\left(x^s - y^s\right), \qquad \overset{2}{\delta} = \delta_{(3)}\left(x^s - z^s\right),$$

which is equivalent to (7). Evaluating $\underline{T}^{\alpha\beta}{}_{;\beta} = 0$ we obtain the equations of motion, which for the first body read

$$\begin{aligned} &\overset{1}{t}{}^{\alpha 0}{}_{,0} + \overset{1}{t}{}^{00}\left\{\begin{matrix}\widetilde{\alpha}\\ 00\end{matrix}\right\} + 2\,\overset{1}{t}{}^{0r}\left\{\begin{matrix}\widetilde{\alpha}\\ 0r\end{matrix}\right\} + \overset{1}{t}{}^{rs}\left\{\begin{matrix}\widetilde{\alpha}\\ rs\end{matrix}\right\} + \overset{1}{t}{}^{00r}\left\{\begin{matrix}\widetilde{\alpha}\\ 00\end{matrix}\right\}_{,r} \\ &\qquad + 2\,\overset{1}{t}{}^{0sr}\left\{\begin{matrix}\widetilde{\alpha}\\ 0s\end{matrix}\right\}_{,r} + \overset{1}{t}{}^{str}\left\{\begin{matrix}\widetilde{\alpha}\\ st\end{matrix}\right\}_{,r} = 0, \end{aligned} \tag{12}$$

$$\overset{1}{t}{}^{\alpha 0r}{}_{,0} + \overset{1}{t}{}^{\alpha 0}\dot{y}^r - \overset{1}{t}{}^{\alpha r} + \overset{1}{t}{}^{00r}\left\{\begin{matrix}\widetilde{\alpha}\\ 00\end{matrix}\right\} + 2\,\overset{1}{t}{}^{0sr}\left\{\begin{matrix}\widetilde{\alpha}\\ 0s\end{matrix}\right\} + \overset{1}{t}{}^{str}\left\{\begin{matrix}\widetilde{\alpha}\\ st\end{matrix}\right\} = 0, \tag{13}$$

$$\overset{1}{t}{}^{\alpha 0r}\dot{y}^s + \overset{1}{t}{}^{\alpha 0s}\dot{y}^r - \overset{1}{t}{}^{\alpha sr} - \overset{1}{t}{}^{\alpha rs} = 0, \qquad \dot{y}^r = dy^r/dt. \tag{14}$$

The equations for the second body are similar. Assuming the vanishing of $\overset{1}{t}{}^{00r}$ and $\overset{2}{t}{}^{00r}$ (cf. the preceding section), one obtains from (14) for $\alpha = 0$:

$$\overset{1}{S}{}^{rs} \overset{\text{def}}{=} 2\,\overset{1}{t}{}^{0rs} = -\,\overset{1}{S}{}^{sr}$$

and for $\alpha = t$:

$$\overset{1}{t}{}^{str} = \tfrac{1}{2}\left(\overset{1}{S}{}^{sr}\dot{y}^t + \overset{1}{S}{}^{tr}\dot{y}^s\right).$$

The field equations can be solved by the EIH method. Denoting $t^{00} = m$ the equation for $\underset{2}{g}{}_{00}$ can be written as

$$\Delta \underset{2}{g}_{00} = 8\pi \left(\overset{1}{\underset{2}{m}} \overset{1}{\delta} + \overset{2}{\underset{2}{m}} \overset{1}{\delta} \right).$$

The solution of this equation was given in lecture II. As before, one can take $\underset{2}{g}_{ik} = \delta_{ik} \underset{2}{g}_{00}$. The equations for $\underset{3}{g}_{0k}$ and $\underset{4}{g}_{00}$ contain new terms, due to rotation of the bodies (the angular momentum S^{rs} is of the third order and does not enter the second order equations). The "rotating contributions" to $\underset{3}{g}_{0k}$ and $\underset{4}{g}_{00}$ have respectively the form $2\mathbf{S} \times \operatorname{grad} r^{-1}$ and $2\mathbf{v} \left(\mathbf{S} \times \operatorname{grad} r^{-1}\right)$ where $\mathbf{S}$ is the vector associated with the skew tensor S^{rs} and $\mathbf{v}$ is the velocity of the body.

It is now possible to expand (12)–(14) into power series and obtain equations for m, S and y. Equation (12) for $\alpha = 0$ gives in the third order the conservation of $\underset{2}{m}$ and in the fifth order determines $\underset{4}{m}$. Equation (13) for $\alpha = 0$ determines $\underset{3}{t}^{\,0r}$ and $\underset{5}{t}^{\,0r}$, and for $\alpha = s$ gives

$$\overset{1}{\underset{4}{t}}{}^{sr} = \overset{1}{\underset{2}{m}}\, \dot{y}^s \dot{y}^r \qquad \text{and} \qquad \overset{1}{\underset{3}{S}}{}^{sr}{}_{,0} = 0. \tag{15}$$

Equation (12) for $\alpha = k$ gives in the fourth order the Newtonian equations of motion. To the sixth order this equation gives *the post-Newtonian equation of translatory motion* [53]:

$$\begin{aligned} \overset{1}{m}\, \ddot{y}^k - \overset{1}{m}\overset{2}{m}(r^{-1})_{,k} = \overset{1}{m}\overset{2}{m} \Big[& \Big(\dot{y}^s \dot{y}^s + \tfrac{3}{2} \dot{z}^s \dot{z}^s - 4 \dot{y}^s \dot{z}^s - 4 \overset{2}{m}\, r^{-1} - 5 \overset{1}{m}\, r^{-1} \Big) \left(r^{-1}\right)_{,k} \\ & + \left(4 \dot{y}^s \left(\dot{z}^k - \dot{y}^k\right) + 3 \dot{y}^k \dot{z}^s - 4 \dot{z}^k \dot{z}^s \right) \left(r^{-1}\right)_{,s} + \tfrac{1}{2} \dot{z}^m \dot{z}^n r_{,kmn} \Big] \\ & + \Big[\overset{2}{m}\, \overset{1}{S}{}^{rs} \left(2\dot{z}^s - \dot{y}^s\right) + \overset{1}{m}\, \overset{2}{S}{}^{rs} \left(\dot{z}^s - 2\dot{y}^s\right) \Big] \left(r^{-1}\right)_{,kr} \\ & + 2 \Big(\overset{2}{m}\, \overset{1}{S}{}^{kr} + \overset{1}{m}\, \overset{2}{S}{}^{kr} \Big) \left(\dot{z}^s - \dot{y}^s\right) \left(r^{-1}\right)_{,rs}. \end{aligned} \tag{16}$$

$\overset{1}{m}$ and $\overset{2}{m}$ denote here the *second* order masses; the subscript 3 under S has also been omitted; $r = |\mathbf{y} - \mathbf{z}|$ is the distance between the bodies and $r_{,k} = \partial r / \partial y^k$. The interpretation of $\underset{3}{S}{}^{rs}$ as the internal angular momentum is justified by the formula

$$\underset{3}{S}{}^{rs} = \int \left[\left(x^s - y^s\right) \underset{3}{T}{}^{0r} - \left(x^r - y^r\right) \underset{3}{T}{}^{0s} \right] dV.$$

This Newtonian angular momentum is conserved by virtue of (15) and it introduces some relativistic corrections to the motion in the 6th order. Terms of order l^2/L^2 (quadratic in the S) have been neglected in (16) as small compared with l/L.

In order to integrate the equations (16) it is convenient to put them in the Lagrange form. A Lagrangian function for non-rotating bodies has been found by Fichtenholz [24] and by Infeld [37] and the terms corresponding to corrections due to rotation were obtained by Tulczyjew [53]. The symmetry properties of the Lagrangian allow us to write some first integrals of the equations of motion.

We now quote some results under the simplifying assumptions that the mass of the second body is much larger than that of the first ($M = \overset{2}{m} \gg \overset{1}{m} = m$) and that only the second body has an internal angular momentum $\mathbf{S}$[8]. Introducing the vector

$$\mathbf{J} = m\left(1 + \tfrac{1}{2}v^2 + 3M/r\right)(\mathbf{r}\times\mathbf{v}) + 2m\mathbf{r}\times(\mathbf{r}\times\mathbf{S})r^{-3}, \tag{17}$$

where $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{r}$ is the radius-vector pointing from the second body to the first, one can derive the following equation

$$\frac{d\mathbf{J}}{dt} = \frac{2}{r^3}\mathbf{S}\times\mathbf{J}. \tag{18}$$

In the Newtonian approximation $\mathbf{J}$ is simply the (orbital) angular momentum of the first particle and it is conserved by virtue of (18). The absolute value of $\mathbf{J}$ is conserved even in the next approximation, however the vector $\mathbf{J}$ itself precesses around the constant vector $\mathbf{S}$. For an orbit which is circular in the Newtonian approximation ($r = R =$ const.) the angular velocity of precession is equal to

$$2\mathbf{S}R^{-3} = \mathbf{const}.$$

If the Newtonian motion takes place in a plane perpendicular to $\mathbf{S}$, then $\mathbf{J} = \mathbf{const}$ and the post-Newtonian motion is plane too. In this case the trajectory of the particle is a "rotating ellipse" and the advance of periastron per one revolution is given by

$$\Delta\psi = \frac{6\pi M}{p}\left(1 - \frac{4}{3}\frac{mS}{MJ}\right), \tag{19}$$

where p is the semi-latus-rectum of the ellipse. For $S = 0$ this formula reduces to the usual expression for the advance of the perihelion.

[8] Tulczyjew's original work deals with the general case of two rotating, heavy bodies.

For an artificial satellite moving near the Earth, the advance of the perigee due to rotation of the Earth is equal to 53” per century [58]. The angular velocity of precession for such a satellite is equal to 26” per century.

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[14] Name of publisher: Gosudarstvennoe Izdatel'stvo Tekhniko-Teoreticheskoy Literatury. English translation of the book, under the same title, was published by Pergamon Press, New York 1959.

[15] This reference is incorrect. It should read: N. M. Petrova, *Zh. Eksper. Teor. Fiz.* **19**, 989 (1949).

[16] This reference is incorrect. It should read: I. G. Fichtenholz, *Zh. Eksper. Teor. Fiz.* **20**, 233 (1950).

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